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Quantum Mechanics as a Matrix Symplectic Geometry

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The main purpose of this work is to describe the quantum analog of the usual classical symplectic geometry and then to formulate quantum mechanics as a noncommutative symplectic geometry. First, we describe a discrete Weyl–Schwinger realization of the Heisenberg group and we develop a discrete version of the Weyl–Wigner–Moyal formalism. We also study the continuous limit and the case of higher degrees of freedom. In analogy with the classical case, we present the noncommutative (quantum) symplectic geometry associated with the matrix algebra $M_N(C)$ generated by the Schwinger matrices.

1. INTRODUCTION AND MOTIVATION

It is well established now that it is possible to give a complete description of Hamiltonian mechanics in the context of *Poisson symplectic geometry* (Guillemin and Sternberg, 1984; Abraham and Marsden, 1985; Arnold, 1989). On the other hand, since the appearance of quantum mechanics, several attempts have been made to give a precise interpretation of the *quantization phenomenon*. This latter refers substantially to the construction of a *quantum system* which admits the classical one as its limit when Planck's constant $\hbar$ tends to zero.

In the beginning, quantum mechanics was interpreted as a statistical theory over phase space. Founding upon the Weyl (1931) quantization procedure Wigner (1932) gave an expression for a phase space distribution function (see also Baker, 1958; Agarwal and Wolf, 1970; Galetti and Toledo Piza, 1988).

Interesting developments of the Weyl–Wigner approach are due to Moyal (1949 who introduced the *sine-Poisson bracket* (or *Moyal bracket*), for func-

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tions on phase space that first corresponds to the commutator of quantum mechanical operators associated to these functions, and second goes into the usual Poisson bracket at the *classical limit*: $\hbar \to 0$ (see also Rivier, 1951; Jordan and Sudarshan, 1961; Remler, 1975; Liu, 1976; Sharan, 1979).

It was realized that the Weyl–Wigner–Moyal quantization procedure can be fit into the context of the deformation theory of algebraic structures (Vey, 1974; Flato *et al.*, 1976; Bayen *et al.*, 1977, 1978; Lichnerowicz, 1983). Recently there has been a revival of this technique of *quantization by deformation* (Bakas *et al.*, 1987; Dunne, 1988; Dunne *et al.*, 1988; Fairlie *et al.*, 1989; Bakas, 1989; Carinena *et al.*, 1989; Gurevich and Rubstov, 1992; Grabowski, 1992; Ballesteros *et al.*, 1992; Kammerer and Valton, n.d.).

In the context of C*-algebras, the set $C^\infty(M)$ of smooth functions on the classical phase space (CPS) M^{2n} of some classical dynamical system forms a *commutative* associative Poisson–Lie C*-algebra equipped with a pointwise product $\cdot$ and with a Poisson bracket $\{\,,\,\}$. The underlying differential geometry is endowed with a (*classical*) *commutative symplectic structure*. So, *quantization* of a classical dynamical system appears in this framework as a *breakdown* of the *commutativity symmetry* of the classical C*-algebra. In fact, the *quantization by deformation* procedure consists in maintaining the structure of the CPS and changing the algebraic structure defined on it by deforming the pointwise product and the Poisson bracket into a *star-product* $*_\hbar$ and a *Moyal bracket* $\{\,,\,\}_\hbar$, respectively, making use of the Weyl correspondence.

For instance, the *noncommutative tori* (Rieffel, 1988) can be considered as *deformation quantizations* of ordinary tori for an appropriate Poisson structure (Rieffel, 1989). It was shown that the action (by translations) of an ordinary torus $T^d \sim \mathbb{R}^d/\mathbb{Z}^d$ on the *noncommutative torus* $A = C^\infty(T^d)$ makes this latter a *noncommutative symplectic manifold* with a *smooth differentiable structure* on which Connes (1980, 1986, 1990) showed how to extend the apparatus of the usual differential geometry involving connection, curvature, Chern classes, etc.

Therefore, the starting point of the *noncommutative differential* (*symplectic*) *geometry* consists in replacing the abstract commutative C*-algebra $C^\infty(M)$ of smooth functions on a *commutative* (*symplectic*) manifold M by a *noncommutative* C*-algebra A of *functions* on a *noncommutative* (*symplectic*) space. It is clear that, in the *noncommutative symplectic* case, one must define the *noncommutative* (or *quantum*) analog of the *commutative* (or *classical*) symplectic structure such that one arrives at the latter from the former by means of a *commutative limit*.

In addition to Connes' (1980, 1986, 1990) approach, there exists another one due to Dubois-Violette (1988, 1990; Dubois-Violette *et al.*, 1989a,b, 1990a,b) which differs from the first one essentially in the definition of the *noncommutative* generalization Ω of the differential algebra of differential forms (for review see also Djemai, n.d.-c). In this approach, a *noncommutative* symplectic structure for A is an element ω of $\Omega^2_{\mathrm{Der}}(A)$, where $\Omega_{\mathrm{Der}}(A)$ is the smallest differential subalgebra of the complex $C(\mathrm{Der}(A); A)$ and $\mathrm{Der}(A)$ is the Lie algebra of all derivations of A. The element ω must satisfy the following conditions:

(i) For a given $H \in A$, there is a unique derivation $\mathrm{ham}(H) \in \mathrm{Der}(A)$ such that

$$\omega(X, \mathrm{ham}(H)) = X(H) \qquad \text{for any} \quad X \in \mathrm{Der}(A)$$

(ii) ω is closed.

It is easy to see that the existence of the *commutative limit* is ensured.

The main aim of this work is to develop a matrix Hamiltonian formalism on a (*torus*) *lattice quantum phase space* using the noncommutative differential geometry of the matrix algebra $M_N(\mathbb{C})$ generated by a privileged basis. We also give another conformation of the fact that quantum mechanics (QM) can be understood as a (*matrix*) *noncommutative symplectic geometry.*

The paper is organized as follows. In Section 2, we recall briefly some notions of classical symplectic geometry. In Section 3, we review the Weyl–Wigner–Moyal formalism. In Section 4, we present the Weyl–Schwinger realization of the Heisenberg group as an extension of the Abelian double cyclic group $Z_N \times Z_N$ and discuss the importance of the choice of Schwinger basis. It appears that one may study physical situations with various numbers of degrees of freedom based upon the prime decomposition of N. We also study the continuous limit $N \to \infty$. Considering the Schwinger basis as a Fourier basis, one may easily see that quantization is deeply tied to the Fourier analysis. In this context, we construct the discrete version of the Weyl–Wigner–Moyal formalism and present explicitly the case of $N = 2$. The apparatus of the noncommutative differential geometry of matrix algebras is presented in Section 5. In Section 6, we describe our matrix Hamiltonian formalism resulting from the use of the Schwinger basis, in complete analogy with the classical case. Finally, Section 7 is devoted to some concluding remarks and perspectives.

2. CLASSICAL SYMPLECTIC FORMALISM

Let us consider a dynamical system evolving in a classical phase space $M^{2d} = T^*(B^n)$ with local coordinates $\epsilon^a = (q^i, p_j)$, $a = 1, \ldots, 2d$, $i, j = 1, \ldots, d$, in such a way that they obey the fundamental Poisson brackets (Guillemin and Sternberg, 1984; Abraham and Marsden, 1985; Arnold, 1989):

$$\{q^i, q^j\}_P = \{p_i, p_j\}_P = 0, \qquad \{q^i, p_j\}_P = \delta^i_j \tag{1}$$

which is compactly summarized as

$$\{\epsilon^a, \epsilon^b\} = \omega^{ab} \tag{2}$$

where the antisymmetric matrix ω^{ab} given by

$$\omega^{ab} = \begin{pmatrix} 0_{d\times d} & 1_{d\times d} \\ -1_{d\times d} & 0_{d\times d} \end{pmatrix} \tag{3}$$

is the inverse of the symplectic matrix ω_{ab}, i.e.,

$$\omega_{ab}\cdot\omega^{bc} = \delta^o_a \tag{4}$$

In fact, ω_{ab} represents the components of the closed nondegenerate symplectic 2-form ω:

$$\omega = \tfrac{1}{2}\omega_{ab} d\epsilon^a \wedge d\epsilon^b = d\theta = dp_i \wedge dq^i \tag{5}$$

where $\theta = p_i dq^i$ is the *canonical* (*Liouville*) *1-form*. This defines the Hamiltonian structure on the phase space M.

Indeed, with any smooth function $H(q, p) = H(\epsilon)$ on M one associates the Hamiltonian vector field

$$X_H = X^b_H \partial_b \tag{6}$$

where $\partial_b = \partial/\partial\epsilon^b$. The components X^b_H, which are given by

$$X^b_H = (\partial_a H(\epsilon))\cdot\omega^{ab} \tag{7}$$

permit us to deduce the Hamiltonian canonical equations

$$\dot{\epsilon}^a = X^a_H(\epsilon) \tag{8}$$

It is easy to see that the Poisson bracket involving two arbitrary functions $F(\epsilon)$ and $G(\epsilon)$ on M is given by

$$\begin{aligned}\{F(\epsilon), G(\epsilon)\}_P &= -\omega(X_F, X_G) = \omega(X_G, X_F) = X_F(G) = -X_G(F) \\ &= (\partial_a F(\epsilon))\cdot\omega^{ab}\cdot(\partial_b G(\epsilon)) = \frac{\partial F}{\partial q^i}\cdot\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\cdot\frac{\partial G}{\partial q^i}\end{aligned} \tag{9}$$

Hence, the set $C^\infty(M)$ of *classical observables* possesses the structure of a Poisson–Lie algebra $A_0 = (C^\infty(M), \cdot, \{\ ,\ \}_P)$ equipped with two internal laws, the pointwise product $\cdot$ and Poisson bracket $\{\ ,\ \}_P$, in such a way that the Jacobi identity

$$\{\{F, G\}_P, h\}_P + \{\{H, F\}_P, G\}_P + \{\{G, H\}_P, F\}_P = 0 \tag{10}$$

is equivalent to the relation

$$[X_F, X_G] = X_{\{F,G\}_P} \tag{11}$$

and the Leibnitz rule

$$\{F, G\cdot H\}_P = \{F, G\}_P\cdot H + G\cdot\{F, H\}_P \tag{12}$$

guarantees that the Hamiltonian vector field is a derivation:

$$X_F(G\cdot H) = X_F(G)\cdot H + G\cdot X_F(H) \tag{13}$$

3. WEYL–WIGNER–MOYAL FORMALISM

Let $B = \mathrm{R}^d$ be a configuration space on which a one-particle system moves and let $M^{2d} = T^*(B) \sim \mathrm{R}^{2d}$ be its associated classical phase space (CPS) with local coordinates (q^i, p_j), $i, j = 1, \ldots, d$. The CPS is then equipped with a Liouville 1-form $\theta = p_i dq^i$ and a classical symplectic structure defined by the closed nondegenerate 2-form $\omega = dp_i \wedge dq^i$.

The *Weyl map* consists in associating to a classical observable $F(\vec{q}, \vec{p}) \in A_0 = (C^\infty(M), \cdot, \{\ ,\ \}_P)$ a *quantum observable* $O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}})$ acting as an operator on the *quantum mechanical Hilbert space* $L^2(B)$, by means of the following operator Fourier transform (Weyl, 1931):

$$O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}}) = \frac{1}{(2\pi)^{2d}} \int_{\mathrm{R}^{4d}} d\vec{a}\, d\vec{b}\, d\vec{p}\, d\vec{q}\, F(\mathbf{q}, \mathbf{p}) \times \exp\{i[\vec{a}\cdot(\vec{\mathbf{p}} - \vec{p}) - \vec{b}\cdot(\vec{\mathbf{q}} - \vec{q})]\} \tag{14}$$

where $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ are the self-adjoint operators on $L^2(B)$ obtained by the correspondence principle from the classical variables $\vec{p}$ and $\vec{q}$. The operators $\vec{\mathbf{p}}$ and $\vec{\mathbf{q}}$ generate the *noncommutative fundamental Heisenberg algebra*:

$$[\mathbf{q}^i, \mathbf{q}^j] = [\mathbf{p}_i, \mathbf{p}_j] = 0, \qquad [\mathbf{q}^i, \mathbf{p}_j] = i\hbar\delta^i_j 1 \tag{15}$$

which is the *quantum* version of the algebra (1). In fact, the quantum mechanical Hilbert space represents the so-called *quantum phase space* (QPS).

The Weyl map

$$F(\vec{q}, \vec{p}) \to O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}}) \tag{16}$$

in invertible, i.e., there is a 1-to-1 correspondence between functions on CPS and their analogs on QPS. In fact, the functions $F(\vec{q}, \vec{p})$, which are often called *Wigner functions*, are defined as ordinary Fourier transforms of the so-called *Wigner densities* $D_F(\vec{a}, \vec{b})$ (Wigner, 1932):

$$F(\vec{q}, \vec{p}) = \frac{1}{(2\pi)^d} \int_{\mathrm{R}^{2d}} d\vec{a}\, d\vec{b}\, D_F(\vec{a}, \vec{b}) \exp\{i[\vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{q}]\} \tag{17}$$

The Wigner functions F, which are o-numbers, are also called *symbols* of the associated operators O_F (Berezin, 1980). If F (respectively $\tilde{F}$) denotes the usual Fourier transform (respectively the operator Fourier transform), we get

$$O_F = \tilde{F}[F^{-1}[F]] = \tilde{F}[D_F] \tag{18a}$$

$$F = F[D_F] = F[\tilde{F}^{-1}[O_F]] \tag{18b}$$

The densities D_F include commonly Dirac deltas or/and their derivatives. This is for the Weyl–Wigner transformation.

Now we will discuss the developments of this approach due to Moyal (1949). It is well known that the differential geometric structure of a manifold M is perfectly determined by the properties of the algebra $A_0 = C^\infty(M)$ of smooth functions on M. Hence, to describe correctly the classical Hamiltonian mechanics one must study the *symplectic differential geometry* of the CPS, or equivalently, the Poisson–Lie algebra $A_0 = (C^\infty(M), \cdot, \{\,,\}_P)$ of *commutative classical observables* on M.

Now, to describe *quantum mechanics* it is straightforward to think of a *noncommutative* generalization of the above geometry. This idea was perceived at the advent of quantum mechanics (Heisenberg, 1925; Born and Jordan, 1925; Born *et al.*, 1926), where this latter appeared as included in the framework of a *noncommutative* version of the notion of Poisson manifold (Dirac, 1926), which will represent the QPS.

In any case, the algebra A of *quantum observables* should be equipped with two internal laws that can be compared with $\cdot$ and $\{\,,\}_P$ in A_0. Since in the statistical Weyl–Wigner formulation of quantum mechanics one does not manipulate operators but their *symbols*, one may think of a *twisted* version of the two internal operations of A_0. The simplest choice consists in *deforming* the *commutative* product $\cdot$ and the Poisson bracket $\{\,,\}_P$ into a *noncommutative* product $*_\nu$ and a *twisted* Poisson bracket $\{\,,\}_\nu$, respectively, by means of a deformation parameter ν such that, for a particular value ν_0 of ν, one has

$$\lim_{\nu \to \nu_0} F *_\nu G = F \cdot G \tag{19a}$$

$$\lim_{\nu \to \nu_0} \{F, G\}_\nu = \{F, G\}_P \tag{19b}$$

In our context, the deformation parameter is no other than Planck's constant $\hbar$ and the *classical limit* $\hbar \to D$ guarantees the passage from the *twisted* algebra $A_\hbar(C^\infty(M), *_\hbar, \{\,,\}_\hbar)$ to the classical one A_0.

The first example of a *star-product* was given by Moyal (1949). Starting from the Weyl map (16), consider the product of two operators $O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}})$ and $O_G(\vec{\mathbf{q}}, \vec{\mathbf{p}})$ and try to define the resulting (deformed) product of the associated symbols F and G. Then, from

$$O_F \circ O_G(\vec{\mathbf{q}}, \vec{\mathbf{p}}) = O_H(\vec{\mathbf{q}}, \vec{\mathbf{p}}) \tag{20}$$

and using (15) and the Glauber formula

$$e^A \cdot e^B = e^{A+B} \cdot e^{[A,B]/2} \tag{21}$$

we get in terms of Wigner densities the following formula:

$$D_H(\vec{a}'', \vec{b}'') = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} d\vec{a}\; d\vec{b}\; D_F(\vec{a}, \vec{b}) \cdot D_G(\vec{a}'' - \vec{a}, \vec{b}'' - \vec{b}) \times \exp\{i\hbar[\vec{a}'' \cdot \vec{b} - \vec{b}'' \cdot \vec{a}]/2\} \tag{22}$$

Knowing that the product $\cdot$ is tied to the *convolution product* $\times$ by

$$F[F \times G] = [F] \cdot F[G] \tag{23a}$$

or,

$$F \cdot G = F[F^{-1}[F] \times F^{-1}[G]] \tag{23b}$$

where

$$(F \times G)(x) = \frac{1}{(2\pi)^{1/2}} \int dy\, F(y) \cdot G(x - y) \tag{23c}$$

then it is possible to identify equation (22) with a *twisted convolution product*:

$$D_H = D_F \times_\hbar D_G \tag{24}$$

From equation (24), one naturally deduces a relation between the *star-product* $*_\hbar$ and the *twisted convolution product* $\times_\hbar$ analogous to (23b):

$$H = F *_\hbar G = F[F^{-1}[F] \times_\hbar F^{-1}[G]] \tag{25}$$

The *Moyal product* is a star-product $*_\hbar$ defined on the space $C^\infty(M, \hbar)$ of formal series in $\hbar$ with coefficients in $C^\infty(M)$ such that

$$F *_\hbar G = F(\vec{q}, \vec{p}) \cdot e^{i\hbar P/2} \cdot G(\vec{q}, \vec{p}) = \sum_{n=0}^{\infty} \frac{(\frac{1}{2} i\hbar)^n}{n!} P^n(F, G) = F(\epsilon) \cdot \exp(\tfrac{1}{2} i\hbar \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b) \cdot G(\epsilon) \tag{26}$$

where $\partial_a = \partial/\partial\epsilon^a$, $\epsilon^a = (q^i, p_j)$ with $a = 1, \ldots, 2d$, $i, j = 1, \ldots, d$; ω^{ab} is the inverse of the symplectic matrix [see (3)] and P is an operator defined by

$$P^0(F, G) = F \cdot G \tag{27a}$$

$$P(F, G) = \{F, G\}_P \tag{27b}$$

$$P^n(F, G) = (-1)^n P^n(G, F) = (\partial_{a_1} \cdots \partial_{a_n} F) \cdot \omega^{a_1 b_1} \cdots \omega^{a_n b_n} \cdot (\partial_{b_1} \cdots \partial_{b_n} G) \tag{27c}$$

$$P^n(F, G) = 0 \qquad \text{for } n > 0 \quad \text{if } F \text{ or } G \text{ is a constant in } C^\infty(M, \hbar) \tag{27d}$$

$$\sum_{\substack{n+m=t \\ t\geq 0}} P^n(P^m(F, G), H) = \sum_{\substack{n+m=t \\ t\geq 0}} P^n(F, P^m(G, H)) \tag{27e}$$

Equation (27e) expresses the associativity of the noncommutative algebra $C^\infty(M, \hbar)$. Moreover, it is easy to see that the Moyal product defined by equation (26) satisfies the axiom (19a). In fact, by means of the precise choice (27c) of suitable forms of the bidifferential operators

$$P^n\colon \quad C^\infty(M, \hbar) \times C^\infty(M, \hbar) \to C^\infty(M, \hbar)$$

that satisfy (27a), (27b), (27d), and (27e), the Moyal product is completely defined.

Now, to emphasize the Poisson–Lie algebra structure of $(C^\infty(M, \hbar), *_\hbar)$, one must define a *deformed Poisson bracket* $\{\ ,\ \}_\hbar$ obeying the axiom (19b).

The *Moyal bracket* is deduced from the commutator of two operators:

$$[O_F, O_G](\vec{\mathbf{q}}, \vec{\mathbf{p}}) =: i\hbar O_{\{F,G\}_\hbar}(\vec{\mathbf{q}}, \vec{\mathbf{p}}) = i\hbar O_H(\vec{\mathbf{q}}, \vec{\mathbf{p}}) \tag{28}$$

Then,

$$\begin{aligned} H(\epsilon) &=: \{F(\epsilon), G(\epsilon)\}_\hbar \\ &=: \frac{1}{i\hbar} [F(\epsilon) *_\hbar G(\epsilon) - G(\epsilon) *_\hbar F(\epsilon)] \\ &= \frac{2}{\hbar} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{\hbar}{2}\right)^{2n+1} P^{2n+1}(F, G) \\ &= \frac{2}{\hbar} \sin\left[\frac{\hbar}{2} P(F, G)\right] \\ &= \frac{2}{\hbar} F(\epsilon) \cdot \sin\left(\frac{\hbar}{2} \overleftarrow{\partial}_a \omega^{ab} \overrightarrow{\partial}_b\right) \cdot G(\epsilon) \end{aligned} \tag{29}$$

It is clear that the axiom (19b) is fulfilled. Moreover, this deformed bracket obeys the Jacobi identity:

$$\{\{F, G\}_\hbar, H\}_\hbar + \{\{H, F\}_\hbar, G\}_\hbar + \{\{G, H\}_\hbar, F\}_\hbar = 0 \tag{30}$$

It also defines a derivation of the Poisson–Lie algebra $(C^\infty(M, \hbar), *_\hbar, \{\ ,\ \}_\hbar)$ with respect to $*_\hbar$, i.e., it obeys the Leibnitz rule:

$$\{F, G *_\hbar H\}_\hbar = \{F, G\}_\hbar *_\hbar H + G *_\hbar \{F, H\}_\hbar \tag{31}$$

Finally, one has the following correspondences:

$$\begin{array}{ccccc} \text{operator} & & \text{classical} & & \text{star-deformed} \\ \circ & \xleftarrow{\quad\textit{canonical quantization}\quad} & \cdot & \xrightarrow{\quad\textit{Weyl–Wigner–Moyal quantization}\quad} & *_{\hbar} \\ \frac{1}{i\hbar}[\,,\,] & \longleftarrow & \{\,,\,\}_{\mathrm{P}} & \longrightarrow & \{\,,\,\}_{\hbar} \end{array}$$

4. WEYL–SCHWINGER REALIZATION OF THE HEISENBERG GROUP

If we represent the Heisenberg group as an extension of the Abelian double cyclic group $Z_N \otimes Z_N$, then finite representations of this group which are realized by two unitary operators U and V satisfying the basic relation

$$V \cdot U = \omega U \cdot V \tag{32}$$

with ω a complex number, are obtained by taking $N \times N$ matrices for U and V such that

$$U^N = 1_{N\times N} = V^N \tag{33}$$

By taking determinants in (32), it follows that

$$\omega = \exp[(i/\hbar)(2\pi\hbar/N)] = \exp(i2\pi/N) \tag{34}$$

For each integer number N, there is one realization of the Heisenberg group. Let $\{ |\alpha_k\rangle, k \in Z\}$ be a basis of orthonormalized kets for the space of states. Assume that the $|\alpha_k\rangle$ are eigenkets of the operator V:

$$V|\alpha_k\rangle = v_k|\alpha_k\rangle / v_k = \omega^k \tag{35a}$$

$$V^m|\alpha_k\rangle = \omega^{km}|\alpha_k\rangle \Rightarrow V^m = \sum_{k=0}^{N-1} \omega^{km}|\alpha_k\rangle\langle\alpha_k| \tag{35b}$$

Then U is defined by

$$U|\alpha_k\rangle = |\alpha_{k+1}\rangle \tag{35c}$$

$$U^n|\alpha_k\rangle = |\alpha_{k+n}\rangle \Rightarrow U^n = \sum_{k=0}^{N-1} |\alpha_{k+n}\rangle\langle\alpha_k| \tag{35d}$$

with

$$|\alpha_{k+N}\rangle \equiv |\alpha_k\rangle \tag{35e}$$

where the integers k, m, n are defined modulo N.

These definitions ensure that U and V satisfy (33). In this basis, the matrices V and U are given by

$$V = \begin{pmatrix} 1 & & & \cdots & 0 \\ \cdot & \omega & & & \cdot \\ \cdot & & \omega^2 & & \cdot \\ \cdot & & & \ddots & \cdot \\ 0 & \cdots & & & \omega^{N-1} \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ & \cdot & \cdot & \cdot & 0 \\ 0 & & \cdot & \cdot & 1 \\ 1 & 0 & & \cdots & 0 \end{pmatrix} \tag{35f}$$

In any case, and independently of the choice of the basis, U and V obey (33) and satisfy

$$V^n \cdot U^m = \omega^{mn} U^m \cdot V^n \tag{36}$$

Any element W of the operator algebra of the Heisenberg group can be determined, up to a scalar factor, by a triple of integers (m, n, p) such that

$$W = U^m \cdot V^n \omega^p \tag{37}$$

These monomials in U and V constitute a complete basis for all quantum operators related to some chosen physical system. The above expressions are invariant under the simultaneous transformations

$$U \to V, \qquad V \to U^{-1}, \qquad \text{and} \qquad m \to n, \quad n \to -m \tag{38}$$

So, let us denote by Y_{mn} the operators that are invariant under this symmetry:

$$Y_{mn} = \omega^{mn/2} U^m \cdot V^n \tag{39}$$

These elements satisfy

$$Y_{on} = V^n, \qquad Y_{mo} = U^m, \qquad Y_{oo} = 1, \qquad Y_{mn}^{-1} = Y_{mn}^{+} = Y_{-m,-n} \tag{40}$$

and form a complete orthonormal basis of the group algebra with the following properties:

Associativity:

$$(Y_{mn} \cdot Y_{kl}) \cdot Y_{pq} = Y_{mn} \cdot (Y_{kl} \cdot Y_{pq}) \tag{41a}$$

Quasiperiodicity:

$$Y_{Nn} = (-1)^n Y_{on}, \qquad Y_{mN} = (-1)^m Y_{mo} \tag{41b}$$

Two-foldedness:

$$\Upsilon_{NN} = (-1)^N 1 \tag{41c}$$

Power:

$$(\Upsilon_{mn})^p = \Upsilon_{pm,pn} \tag{41d}$$

In the basis $\{|\alpha_k\rangle, k \in Z_N\}$ where V is diagonal, the operators Υ_{mn} take the form

$$\Upsilon_{mn} = \sum_{k=0}^{N-1} \omega^{n(2k+m)/2} |\alpha_{k+m}\rangle\langle\alpha_k| \tag{42}$$

and its corresponding matrix $N \times N$ is given by

$$\Upsilon_{mn} = \omega^{mn/2} \left(\begin{array}{ccccc|ccccc} & & O_{m\times(N-m)} & & & \omega^{n(N-m)} & 0 & 0 & \cdots & 0 \\ & & & & & 0 & \omega^{n(N-m+1)} & 0 & \cdots & 0 \\ & & & & & 0 & 0 & \omega^{n(N-m+2)} & \cdots & 0 \\ & & & & & \vdots & & & \ddots & \\ & & & & & 0 & & \cdots & & \omega^{n(N-1)} \\ \hline 1 & 0 & 0 & \cdots & 0 & & & & & \\ 0 & \omega^n & 0 & \cdots & 0 & & & & & \\ 0 & 0 & \omega^{2n} & \cdots & 0 & & & O_{(N-m)\times m} & & \\ \vdots & & & \ddots & \vdots & & & & & \\ 0 & & \cdots & & \omega^{n(N-m-1)} & & & & & \end{array} \right) \tag{43}$$

Although the operators U and V were introduced by Weyl (1931), it was Schwinger (1960, 1961, 1970) who described quantum mechanics in this formalism.

Any operator A belonging to the group algebra will be written in the Schwinger basis $\{\Upsilon_{mn}; m, n = 0, \ldots, N-1\}$ as

$$A = \frac{1}{N} \sum_{m,n} a^{mn} \Upsilon_{mn} \tag{44}$$

with complex coefficients

$$a^{mn} = \mathrm{Tr}[\Upsilon^+_{mn} \cdot A] = (\Upsilon^+_{mn} \cdot A)^{oo} \tag{45}$$

and with

$$\mathrm{Tr}[\Upsilon_{mn}] = N\delta_{mo}\delta_{no} \tag{46a}$$

$$\mathrm{Tr}[A] = a^{oo} = A^{oo} \tag{46b}$$

$$\mathrm{Tr}[A^+ \cdot A] = \sum_{m,n} |a^{mn}|^2 \tag{46c}$$

Here, the trace defines on the operator algebra an internal product given by

$$\langle A, B\rangle = \mathrm{Tr}[B^{+}\cdot A] \tag{47}$$

and consequently the metric follows as

$$g_{(mn)(rs)} = \frac{1}{N}\mathrm{Tr}[Y^{+}_{mn}\cdot Y_{rs}] = \delta_{mr}\delta_{ns} \tag{48}$$

In this metric, the complete symmetrized basis $\{Y_{mn}\}$ is orthonormal. Then, the set of operators forms a *metric algebra with unity.*

Henceforth, we will use a more compact notation such that for $m_1, m_2, n_1, n_2, \ldots \in \mathbb{Z}$, one has

$$\vec{m} = (m_1, m_2), \qquad \vec{n} = (n_1, n_2), \qquad \vec{o} = (o, o) \tag{49a}$$
$$\vec{m} + \vec{n} = (m_1 + n_1, m_2 + n_2), \qquad \vec{m}\cdot\vec{n} = m_1 n_1 + m_2 n_2$$

and,

$$\vec{m} \times \vec{n} =: m_1 n_2 - m_2 n_1 \tag{49b}$$

Then,

$$Y_{\vec{m}} = \omega^{m_1 m_2/2} U^{m_1}\cdot V^{m_2} \tag{50a}$$

$$Y_{\vec{o}} = 1, \qquad Y_{\vec{m}}^{-1} = Y_{\vec{m}}^{\pm} = Y_{-\vec{m}} \tag{50b}$$

$$A = \frac{1}{N}\sum_{\vec{m}} a^{\vec{m}} Y_{\vec{m}}, \qquad a^{\vec{m}} = \mathrm{Tr}[Y_{\vec{m}}^{\pm}\cdot A] \tag{50c}$$

$$g_{\vec{m}\vec{n}} = \frac{1}{N}\mathrm{Tr}[Y_{\vec{m}}^{\pm}\cdot Y_{\vec{n}}] = \delta_{\vec{m}\vec{n}} \tag{50d}$$

$$\delta_{\vec{m}\vec{n}} = \delta_{m_1 n_1}\cdot\delta_{m_2 n_2} \tag{50e}$$

For each value of N, the operators $Y_{\vec{m}}$ realize a representation that makes use of a two-cocycle α_2. Occurring frequently in quantum mechanics, these representations are called *projective* representations and the operators $Y_{\vec{m}}$ in them obey a generalized composition law:

$$Y_{\vec{n}}\cdot Y_{\vec{m}} = \{\exp[2i\alpha_2(\vec{m}, \vec{n})]\} Y_{\vec{m}}\cdot Y_{\vec{n}} = \{\exp[i\alpha_2(\vec{m}, \vec{n})]\} Y_{\vec{m}+\vec{n}} \tag{51}$$

with

$$\alpha_2(\vec{m}, \vec{n}) = \frac{\pi}{N}\vec{m}\times\vec{n} = -\alpha_2(\vec{n}, \vec{m}) \tag{52}$$

In general, any two elements $W_1 = U^{m_1} \cdot V^{n_1} \omega^{p_1}$ and $W_2 = U^{m_2} \cdot V^{n_2} \omega^{p_2}$ will satisfy the following Heisenberg group defining relations in terms of triples:

$$(m_1, n_2, p_1) * (m_2, n_2, p_2) = (m_1 + m_2, n_1 + n_2, p_1 + p_2 + \tfrac{1}{2}(m_1 n_2 - m_2 n_1)) \tag{53}$$

From the associativity property (41a), we get the following consistency condition:

$$\Delta\alpha_2 = \alpha_2(\vec{n}, \vec{p}) - \alpha_2(\vec{m} + \vec{n}, \vec{p}) + \alpha_2(\vec{m}, \vec{n} + \vec{p}) - \alpha_2(\vec{m}, \vec{n}) = 0 \quad (\mathrm{mod}(\mathrm{Z})) \tag{54}$$

Quantum mechanics also makes use of the one-cochain. Indeed, the action of the operators $Y_{\vec{m}}$ on a state $|\alpha_k\rangle$ is given by

$$Y_{\vec{m}} |\alpha_k\rangle = \{\exp[i\alpha_1(k; \vec{m})]\} |\alpha_{k+m_1}\rangle \tag{55}$$

with

$$\alpha_1(k; \vec{m}) = \frac{\pi m_2}{N}(2k + m_1) \tag{56}$$

It turns out that the so-called *fundamental* cocycle α_2 is given by

$$\alpha_2 = \Delta\alpha_1 = \alpha_1(k + n_1; \vec{m}) - \alpha_1(k; \vec{m} + \vec{n}) + \alpha_1(k; \vec{n}) \tag{57}$$

where Δ is a nilpotent derivative (coboundary operator) of some cohomology giving information on the projective representation under consideration (Aldrovandi and Galetti, 1990).

In fact, it has been shown in Aldrovandi and Galetti (1990) that α_1 and α_2 given by (56) and (52), respectively, result from the action of algebraic cochains on the group elements

$$\alpha_1(k; \vec{m}) = \alpha_1(k; Y_{\vec{m}}) \tag{58a}$$

$$\alpha_2(\vec{m}, \vec{n}) = \alpha_2(k; Y_{\vec{m}}, Y_{\vec{n}}) \tag{58b}$$

The nilpotent derivative Δ is defined on such cochains. Thus, if α_1 is exact, i.e., $\alpha_1 = \Delta\alpha_0$ with α_0 some 0-cochain, α_1 may be eliminated by adding a phase α_0 to the wavefunctions. When α_2 is exact, it may be eliminated by redefining the operators in such a way that they appear state independent. It turns out actually that $\alpha_2 = \Delta\alpha_1$ [see (57)], so that $\Delta\alpha_2 = 0$ [see (54)]. This means that α_2 is a 2-cocycle called the *fundamental cocycle*. It is clear from (52) and (56) that α_1 depends effectively on the state label k, while α_2 does not.

Moreover, the fundamental cocycle will define a *simplicial symplectic structure* on the *lattice quantum phase space* (LQPS) analogous to the classical one on the usual CPS, while the 1-cochain α_1 will play on LQPS the role of the Liouville canonical form θ on CPS. In fact, LQPS is the *lattice torus* described by the double integer index $\vec{m} = (m_1, m_2)$ that label the operators belonging to the Schwinger basis $\{Y_{\vec{m}}\}$, and in which the area of each elementary lattice is equal to $2\pi/N$, tending to zero when $N \to \infty$.

Therefore, the translations resulting from the action of the operator $Y_{\vec{m}}$ on LQPS may be interpreted as unit multiple elementary gaps in the two basic directions (U, V).

The choice of the symmetrized Schwinger basis $\{Y_{\vec{m}}\}$ for the Weyl realization of the Heisenberg group is based essentially on its many remarkable properties, such as the following:

First, for $N = 2$, the $Y_{\vec{m}}$ reduce to the Pauli matrices [see (43)]:

$$Y_{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 = \sigma_0, \qquad Y_{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

$$Y_{11} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \qquad Y_{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3 \tag{59}$$

For $N \geq 2$, the Schwinger basis is a preferable basis admitting additive quantum numbers. It provides the finest grading of the Lie algebra $gl(N, C)$ (Patera and Zassenhaus, 1988). Second, there are many helpful relations, such as

$$\frac{1}{N} \sum_{\vec{m}} Y_{\vec{m}} \cdot A \cdot Y_{\vec{m}}^{\pm} = (\mathrm{Tr}[A])1 \tag{60a}$$

$$\frac{1}{N} \sum_{k} \omega^{k(m-n)} = \delta^{mn} \tag{60b}$$

$$\frac{1}{N^2} \sum_{\vec{k}} e^{2i\alpha_2(\vec{k}, \vec{m} - \vec{n})} = \delta^{\vec{m}, \vec{n}} \tag{60c}$$

Third, because of the two-foldedness property (41c), the $Y_{\vec{m}}$ constitute a double covering mod(N) of the torus, with (m_1, m_2) playing the role of coordinates mod(N). Here U^{m_1} and V^{m_2} may also be seen as *global coordinates* with values in $Z_N \otimes Z_N$ appearing as *noncommutative point functions* because of the *projective* character of the representation. It was shown in Aldrovandi and Galetti (1990) how closed paths on LQPS lead to open paths on operator space and how this fact is related to *noncommutativity*.

Fourth, one may study physical situations with various numbers of degrees of freedom following the value of N. For instance, when N is a prime number, the pair (U, V) describes one degree of freedom taking on N possible values. If N is not prime, then it is a product of, say, d prime numbers:

$$N = N_1 \times \cdots \times N_j \times \cdots \times N_d \tag{61}$$

and the Schwinger basis becomes a product of d independent subbases, one for each prime factor, that is, one for each degree of freedom:

$$\begin{aligned} \{Y_{mn}\} &= \{\omega^{mn/2} U^m, V^n\} \\ &= \bigotimes_{j=1}^{d} \{Y_{m_j n_j}\} = \bigotimes_{j=1}^{d} \{\omega_j^{m_j n^j/2} U^{m_j} \cdot V^{n^j}\} \end{aligned} \tag{62}$$

where $m, n = 0, 1, \ldots, N - 1$ and $m_j, n^j = 0, 1, \ldots, (N_j - 1)$, and

$$\omega_j = \exp[i(2\pi/N_j)] \tag{63}$$

This result implies a classification of the *quantum degrees of freedom* in terms of prime decomposition of N. If we want to work with two or more degrees of freedom, we must use for N a well-chosen nonprime value.

Fifth, the usual situation of the position $\vec{\mathbf{q}} = (\mathbf{q}^1, \ldots, \mathbf{q}^d)$ and the momentum $\vec{\mathbf{p}} = (\mathbf{p}_1, \ldots, \mathbf{p}_d)$ operators is recovered if we choose

$$Y_{0n^j} = V^{n^j} = \exp[i(2\pi/\hbar N_j)^{1/2} n^j \cdot \mathbf{p}_j] \tag{64a}$$

$$Y_{m_j 0} = U^{m_j} = \exp[i(2\pi/\hbar N_j)^{1/2} m_j \cdot \mathbf{q}^j] \tag{64b}$$

$j = 1, \ldots, d$. Then, using the defining commutation relations of the Heisenberg formulas (15) and the Glauber formula (21), we find for equation (62)

$$Y_{\vec{\mu}} = \exp\left[i \sum_j \left(\frac{2\pi}{\hbar N_j} \right)^{1/2} (n^j \cdot \mathbf{p}_j + m_j \cdot \mathbf{q}^j) \right] \tag{65}$$

where we have used the following compact notation:

$$\vec{\mu} = (\vec{m}, \vec{n}) \tag{66}$$

with $\vec{m} = (m_1, m_2, \ldots, m_d)$ and $\vec{n} = (n^1, n^2, \ldots, n^d)$.

To pass to the continuous case, it is sufficient to take to infinity the torus radii, while $N \to \infty$ if $d = 1$, or $N_j \to \infty$ with $j = 1, \ldots, d$ when N is not a prime number ($d > 1$). We define this limit as follows:

$$(2\pi/\hbar N_j)^{1/2} n^j \xrightarrow[N_j \to \infty]{} a^j \tag{67a}$$

$$(2\pi/\hbar N_j)^{1/2} m_j \xrightarrow[N_j \to \infty]{} -b_j \tag{67b}$$

so that

$$\omega_j^{m_j n^j/2} \underset{N_j\to\infty}{\longrightarrow} \exp\left(\frac{-i\hbar}{2}\, a^j \cdot b_j\right) \tag{67c}$$

$$U^{m_j} \underset{N_j\to\infty}{\longrightarrow} U(b_j) = \exp(-ib_j \cdot \mathbf{q}^j) \tag{67d}$$

$$V^{n^j} \underset{N_j\to\infty}{\longrightarrow} V(a^j) = \exp(ia^j \cdot \mathbf{p}_j) \tag{67e}$$

$$\Upsilon_{\vec{\mu}} \underset{\text{all } N_j\to\infty}{\longrightarrow} \Upsilon(\vec{a}, \vec{b}) = \exp[i(\vec{a}\cdot\vec{\mathbf{p}} - \vec{b}\cdot\vec{\mathbf{q}})] \tag{67f}$$

where $\vec{a} = (a^1, \ldots, a^j, \ldots, a^d)$ and $\vec{b} = (b_1, \ldots, b_j, \ldots, b_d)$ are some constant dual vectors characterizing translations

$$\vec{\mathbf{q}} \to \vec{\mathbf{q}} + \vec{a} \tag{68a}$$

$$\vec{\mathbf{p}} \to \vec{\mathbf{p}} + \vec{b} \tag{68b}$$

affected by the operator $\Upsilon(\vec{a}, \vec{b})$ on the $2d$-dimensional phase space, which is a *Hilbert space* corresponding to the true *quantum phase space* (QPS).

The elements $\Upsilon(\vec{a}, \vec{b})$ obey a generalized composition law analogous to (51):

$$\Upsilon(\vec{c}, \vec{d})\cdot\Upsilon(\vec{a}, \vec{b}) = \exp\{i\alpha_2[(\vec{a}, \vec{b}); (\vec{c}, \vec{d})]\}\Upsilon(\vec{a} + \vec{c}, \vec{b} + \vec{d}) \tag{69}$$

where

$$\begin{aligned}\alpha_2[(\vec{a}, \vec{b}); (\vec{c}, \vec{d})] &= \alpha_2[\Upsilon(\vec{a}, \vec{b}); \Upsilon(\vec{c}, \vec{d})] \\ &= \frac{\hbar}{2}\,(\vec{a}, \vec{b}) \times (\vec{c}, \vec{d}) \\ &= \frac{\hbar}{2}\,(\vec{a}\cdot\vec{d} - \vec{b}\cdot\vec{c})\end{aligned} \tag{70}$$

Here, one must make the following remark. The 2-cocycle α_2 given by equation (52) enables us to define the so-called *simplicial symplectic structure* on the *lattice quantum phase space* (LQPS) in analogy with the ordinary symplectic structure on the CPS. It also reduces to it in the *classical limit.* As for equation (70), it gives its *continuous* analog, which enables us to define the true *quantum symplectic structure* on the *quantum mechanical Hilbert space* (QPS).

Sixth, the Schwinger basis may also be considered as a Fourier basis and then becomes fundamental for the Weyl–Wigner map:

$$O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}}) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} d\vec{a}\, d\vec{b}\; D_F(\vec{a}, \vec{b}) \Upsilon(\vec{a}, \vec{b}) \tag{71}$$

where $D_F(\vec{a}, \vec{b})$ is the Wigner density associated to the symbol $F(\vec{q}, \vec{p})$ of the operator $O_F(\vec{\mathbf{q}}, \vec{\mathbf{p}})$.

This shows that *quantization* is deeply tied to Fourier analysis, since any operator is given as a Fourier expansion.

Notice that the phase α_2 in (69)–(70) plays the role of a quantum correction, which is expressed in terms of the classical Poisson bracket [see (9)]

$$\alpha_2[(\vec{a}, \vec{b}); (\vec{c}, \vec{d})] = \frac{\hbar}{2} \{\vec{a}\cdot\vec{p} - \vec{b}\cdot\vec{q}, \vec{c}\cdot\vec{p} - \vec{d}\cdot\vec{q}\}_{\mathrm{P}} \tag{72}$$

Here, the functions $F(\vec{q}, \vec{p})$ and $G(\vec{q}, \vec{p})$ under consideration are the simplest linear functions on QPS.

Generalizing (44) or (50c) for d degrees of freedom, we define any element A in operator algebra by

$$A = \frac{1}{N^d} \sum_{\vec{\mu}} a^{\vec{\mu}} \Upsilon_{\vec{\mu}} \tag{73a}$$

with

$$\mathrm{Tr}[\Upsilon_{\vec{\mu}}] = N^d \delta_{\vec{\mu}\vec{0}}, \qquad \mathrm{Tr}[A] = a^{\vec{0}}, \qquad a^{\vec{\mu}} = \mathrm{Tr}[\Upsilon_{\vec{\mu}}^{\pm}\cdot A] \tag{73b}$$

For any operator A, one can define its continuous analogy by taking the following continuous limits:

$$a^{\vec{\mu}} \xrightarrow[N\to\infty]{} D_F(\vec{a}, \vec{b}) \tag{74}$$

$$\frac{1}{N^d} \sum_{\vec{\mu}} \xrightarrow[N\to\infty]{} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} d\vec{a}\, d\vec{b} \tag{75}$$

Then, the operator expansion (73a) appears as the *discrete version* of the operator Fourier transform (71):

$$\lim_{N\to\infty} A = O_F(\vec{\mathbf{p}}, \vec{\mathbf{q}}) \tag{76}$$

In order to define completely the *quantum symplectic structure* on QPS, we must first consider the product of two arbitrary operators A and B with coefficients $a^{\vec{\mu}}$ and $b^{\vec{\nu}}$, respectively, and then take the continuous limit.

We may also use a more compact notation for the above expressions in analogy with that used in Section 2. Let $\vec{\epsilon} = (\vec{q}, \vec{p})$ be a set of local coordinates on the CPS [see (2)], $\vec{\epsilon} = (\vec{q}, \vec{p})$ its analog on the QPS, $\vec{\alpha} = (\vec{a}, \vec{b})$, and $\vec{\beta} = (\vec{c}, \vec{d})$. Then, for instance, (67f), (69), (70)–(72), and (71) read

$$\Upsilon(\vec{\alpha}) = \exp[i\vec{\alpha} \times \vec{\epsilon}] \tag{77a}$$

$$\Upsilon(\vec{\beta})\cdot\Upsilon(\vec{\alpha}) = \exp\{i\alpha_2[\vec{\alpha}; \vec{\beta}]\}\Upsilon(\vec{\alpha} + \vec{\beta}) \tag{77b}$$

$$\alpha_2[\vec{\alpha}; \vec{\beta}] = \frac{\hbar}{2}\vec{\alpha} \times \vec{\beta} = \frac{\hbar}{2}\{\vec{\alpha} \times \vec{\epsilon}, \vec{\beta} \times \vec{\epsilon}\}_P \tag{77c}$$

$$O_F(\vec{\epsilon}) = \frac{1}{(2\pi)^d}\int_{\mathbf{R}^{2d}} d\vec{\alpha}\; D_F(\vec{\alpha})\Upsilon(\vec{\alpha}) \tag{77d}$$

Now, the product of two operators is given in the discrete and the continuous versions, respectively, as follows:

$$A \circ B = \frac{1}{N^d}\sum_{\vec{\mu}}\frac{1}{N^d}\sum_{\vec{\nu}} a^{\vec{\mu}} b^{\vec{\nu}} \Upsilon_{\vec{\mu}}\cdot\Upsilon_{\vec{\nu}} = C = \frac{1}{N^d}\sum_{\vec{\rho}} c^{\vec{\rho}}\Upsilon_{\vec{\rho}} \tag{78}$$

$$O_F \circ O_G(\vec{\epsilon}) = \frac{1}{(2\pi)^{2d}}\int d\vec{\alpha}\; d\vec{\beta}\; D_F(\vec{\alpha})\cdot D_G(\vec{\beta})\Upsilon(\vec{\alpha})\cdot\Upsilon(\vec{\beta})$$

$$= O_H(\vec{\epsilon}) = \frac{1}{(2\pi)^d}\int d\vec{\sigma}\; D_H(\vec{\sigma})\Upsilon(\vec{\sigma}) \tag{79}$$

Then, the discrete Wigner density $c^{\vec{\rho}}$ and its continuous analog $D_H(\vec{\sigma})$ are given, respectively, by

$$c^{\vec{\rho}} = \frac{1}{N^d}\sum_{\vec{\mu}} a^{\vec{\mu}} b^{\vec{\rho}-\vec{\mu}} \exp[i\alpha_2(\vec{\rho}; \vec{\mu})] \tag{80}$$

$$D_H(\vec{\sigma}) = \frac{1}{(2\pi)^d}\int d\vec{\alpha}\; D_F(\vec{\alpha})\cdot D_G(\vec{\sigma} - \vec{\alpha}) \exp[i\alpha_2(\vec{\sigma}; \vec{\alpha})] \tag{81}$$

In general, a *twisted convolution product* $\times_\nu$ is defined by [see (22)]

$$(f \times_\nu g)(\vec{\sigma}) = \frac{1}{(2\pi)^d}\int d\vec{\alpha}\; f(\vec{\alpha})\cdot g(\vec{\sigma} - \vec{\alpha}) \exp\left(i\frac{\nu}{2}\vec{\sigma} \times \vec{\alpha}\right) \tag{82}$$

Then, in (81) we have a *twisted convolution product* with a *deformation parameter* $\nu = \hbar$ characterizing a "*quantization*" such that the classical situation is recovered when $\hbar \to 0$.

As for (80), it describes also a *twisted convolution product* with a *deformation parameter* $\nu = 2\pi/N$ expressing a "*discretization*" such that the continuous analogy is recovered when $N \to \infty$.

Using (25), one may deduce from the twisted convolution product $\times_\nu$ between Wigner densities a *star-product* $*_\nu$ between symbols.

Knowing that $\alpha_2[\vec{\sigma}; \vec{\alpha}]$ in (81) is given by (77c), then we can define the resulting star-product $*_\hbar$ by

$$F(\vec{\epsilon}) *_\hbar G(\vec{\epsilon}) = F(\vec{\epsilon}) \cdot G(\vec{\epsilon}) + i\frac{\hbar}{2}\{F(\vec{\epsilon}), G(\vec{\epsilon})\}_P + \cdots \tag{83}$$

The resulting algebra equipped with the star-product $*_\hbar$ becomes *non-commutative.* Assumed to be stable under $*_\hbar$, this algebra will concern only functions having compactly supported Fourier transforms such that this star-product may also be defined as the converging expression

$$F *_\hbar G(\vec{\epsilon}) = \frac{1}{(2\pi)^{2d}} \int d\vec{\epsilon}'\, d\vec{\epsilon}''\, F(\vec{\epsilon}')G(\vec{\epsilon}'') \times \exp\{i\hbar[\vec{\epsilon} \times \vec{\epsilon}'' + \vec{\epsilon}'' \times \vec{\epsilon}' + \vec{\epsilon}' \times \vec{\epsilon}]/2\} \tag{84}$$

which is easily deducible from (25) and using the Dirac delta

$$\frac{1}{(2\pi)^{2d}} \int d\vec{\sigma}\, e^{\vec{\sigma} \times \vec{\beta}} = \delta(\vec{\beta}) \tag{85}$$

Before defining the star-product $*_N$, we need to define the "symbol," say $f(\vec{\xi})$, of an operator A (we will denote it A_f in analogy with O_F). In other words, we need to define the discrete version of [see (17)]

$$F(\vec{\epsilon}) = \frac{1}{(2\pi)^d} \int d\vec{\alpha}\, D_F(\vec{\alpha}) \exp(i\vec{\alpha} \times \vec{\epsilon}) = F[D_F](\vec{\epsilon}) \tag{86}$$

The problem here is that Fourier transforms are expansions in unitary irreducible representations and in our case we have projective representations. Recall that, in order to have a Weyl–Schwinger realization of the Heisenberg group, one needs to perform an extension of the Abelian double cyclic group $Z_N \otimes Z_N$, so that the truly unitary representations would be actually related to $Z_N \otimes Z_N$ and not to the Heisenberg group.

Knowing that the discrete Wigner densities always convolute in a twisted way with a deformation parameter $\nu = 2\pi/N$ [see (80)], we may use now a unitary representation and define the discrete version of (86) by

$$f(\vec{\xi}) = \frac{1}{N^d} \sum_{\vec{\mu}} a^{\vec{\mu}} \omega^{\vec{\mu} \times \vec{\xi}} = F[a](\vec{\xi}) \tag{87}$$

where $\vec{\xi} = (\vec{r}, \vec{s}) \in Z_N \otimes Z_N$, $\vec{r} = (r_1, r_2, \ldots, r_d)$, $\vec{s} = (s^1, s^2, \ldots, s^d)$, $\vec{\mu} = (\vec{m}, \vec{n})$ [see (66)], ω is given by (34), and $f(\vec{\xi})$ may be viewed as a function on (the Fourier dual of) LQPS (Aldrovandi, 1993).

The inverse of the *discrete Fourier transform* (87) is then defined by

$$a^{\vec{\mu}} = \frac{1}{N^d} \sum_{\vec{\xi}} f(\vec{\xi}) \omega^{-\vec{\mu} \times \vec{\xi}} = F^{-1}[f](\vec{\mu}) \tag{88}$$

Moreover, if A_f is Hermitian, then $(a^{\vec{\mu}})^* = a^{-\vec{\mu}}$ and the symbol f is real. In fact, the relation [see (73a)]

$$A_f = \frac{1}{N^d} \sum_{\vec{\mu}} a^{\vec{\mu}} Y_{\vec{\mu}} = \tilde{F}[a] \tag{89}$$

represents the discrete version of the operator Fourier transform (77d), so that one has [see equations (18)]

$$A_f = \tilde{F}[F^{-1}[f]] = \tilde{F}[a] \tag{90a}$$

$$f = F[a] = F[\tilde{F}^{-1}[A_f]] \tag{90b}$$

and from (78), (80), we recognize

$$C_h = A_f \circ B_g = \tilde{F}[F^{-1}[h]] = \tilde{F}[F^{-1}[f] \times_N F^{-1}[g]] \tag{90c}$$

$$h = f *_N g = F[F^{-1}[f] \times_N F^{-1}[g]] \tag{90d}$$

Before treating an example of this *discretization by deformation* for the case $N = 2$, let us study the commutators of operators from which one may deduce the deformed Poisson bracket $\{\,,\}_\nu$. The commutators

$$[Y_{\vec{\nu}}, Y_{\vec{\mu}}] = 2i \sin(\alpha_2[\vec{\mu}; \vec{\nu}])\, Y_{\vec{\mu}+\vec{\nu}} \tag{91a}$$

$$[Y(\vec{\beta}), Y(\vec{\alpha})] = 2i \sin(\alpha_2(\vec{\alpha}; \vec{\beta}])\, Y(\vec{\alpha} + \vec{\beta}) \tag{91b}$$

generalize, respectively, to

$$\begin{aligned} [A, B] &= \frac{1}{N^d} \sum_{\vec{\nu}} \frac{1}{N^d} \sum_{\vec{\mu}} a^{\vec{\nu}} b^{\vec{\mu}} [Y_{\vec{\nu}}, Y_{\vec{\mu}}] \\ &= i \frac{2\pi}{N} C = \frac{2\pi i}{N} \frac{1}{N^d} \sum_{\vec{\rho}} c^{\vec{\rho}} Y_{\vec{\rho}} \end{aligned} \tag{91c}$$

$$\begin{aligned} [O_F, O_G](\vec{\epsilon}) &= \frac{1}{(2\pi)^{2d}} \int d\vec{\alpha}\, d\vec{\beta}\, D_F(\vec{\beta}) D_G(\vec{\alpha}) [Y(\vec{\beta}), Y(\vec{\alpha})] \\ &= i\hbar O_H(\vec{\epsilon}) = \frac{i\hbar}{(2\pi)^d} \int d\vec{\sigma}\, D_H(\vec{\sigma}) Y(\vec{\sigma}) \end{aligned} \tag{91d}$$

where

$$c^{\vec{\rho}} = \frac{N}{\pi} \frac{1}{N^d} \sum_{\vec{\nu}} a^{\vec{\nu}} b^{\vec{\rho}-\vec{\nu}} \sin\{\alpha_2[\vec{\rho}; \vec{\nu}]\} \tag{91e}$$

$$D_H(\vec{\sigma}) = \frac{2}{\hbar} \frac{1}{(2\pi)^d} \int d\vec{\beta}\, D_F(\vec{\beta}) D_G(\vec{\sigma} - \vec{\beta}) \sin\{\alpha_2[\vec{\sigma}; \vec{\beta}]\} \tag{91f}$$

represent now the Wigner densities corresponding to the discrete and continous versions, respectively, of the Moyal bracket $\{\,,\}_\nu$, which is defined in general by

$$\{f, g\}_\nu = \frac{1}{i\nu} (f *_\nu g - g *_\nu f) \tag{92}$$

In the continuous case, the deformed bracket is no other than the Moyal bracket defined in Section 2 [see (28)–(29)]:

$$\begin{aligned} H(\vec{\epsilon}) &= \{F(\vec{\epsilon}), G(\vec{\epsilon})\}_\hbar \\ &= \frac{1}{i\hbar} \{F(\vec{\epsilon}) *_\hbar G(\vec{\epsilon}) - G(\vec{\epsilon}) *_\hbar F(\vec{\epsilon})\} \\ &= \frac{2}{\hbar} \sin\left(\frac{\hbar}{2} \{F(\vec{\epsilon}), G(\vec{\epsilon})\}_P\right) \end{aligned} \tag{93a}$$

with the classical limit

$$\lim_{\hbar \to 0} \{F(\vec{\epsilon}), G(\vec{\epsilon})\}_\hbar = \{F(\vec{\epsilon}), G(\vec{\epsilon})\}_P \tag{93b}$$

The Moyal bracket (93a) can also be defined by the following converging relation [see (84)]:

$$H(\vec{\epsilon}) = \frac{2}{\hbar} \frac{1}{(2\pi)^{2d}} \int d\vec{\epsilon}\,'\, d\vec{\epsilon}\,''\, F(\vec{\epsilon}\,') G(\vec{\epsilon}\,'') \sin\left[\frac{\hbar}{2} (\vec{\epsilon} \times \vec{\epsilon}\,'' + \vec{\epsilon}\,'' \times \vec{\epsilon}\,' + \vec{\epsilon}\,' \times \vec{\epsilon})\right] \tag{93c}$$

Using (80) and (87), one can define the discrete versions of (84) and (85) as follows:

$$h(\vec{\xi}) = \frac{1}{N^d} \sum_{\vec{\xi}\,'} \frac{1}{N^d} \sum_{\vec{\xi}\,''} f(\vec{\xi}\,') g(\vec{\xi}\,'') \times \exp\left[i \frac{\pi}{N} (\vec{\xi} \times \vec{\xi}\,'' + \vec{\xi}\,'' \times \vec{\xi}\,' + \vec{\xi}\,' \times \vec{\xi})\right] \tag{94a}$$

$$\frac{1}{N^{2d}} \sum_{\vec{\mu}} \omega^{\vec{\mu}\times\vec{\nu}} = \delta^{\vec{\nu},\vec{0}} \tag{94b}$$

Let us remark that (94b) is a generalization of (60c). Now, one may derive the deformed bracket $\{\,,\}_N$ either using (91e) or directly from (92) and (94a) in analogy with (93c):

$$h(\vec{\xi}) = \frac{N}{\pi}\frac{1}{N^d}\sum_{\vec{\xi}'}\frac{1}{N^d}\sum_{\vec{\xi}''} f(\vec{\xi}')g(\vec{\xi}'')\sin\left[\frac{\pi}{N}(\vec{\xi}\times\vec{\xi}'' + \vec{\xi}''\times\vec{\xi}' + \vec{\xi}'\times\vec{\xi})\right] \tag{94c}$$

Finally, let us treat the case $N = 2$. Here, we are dealing with Pauli matrices $\sigma_1 = Y_{10}$, $\sigma_2 = Y_{11}$, and $\sigma_3 = Y_{01}$ [see (59)] with one degree of freedom ($d = 1$). Using (41b), one may easily see that [see (51)]

$$[Y_{n_1n_2}, Y_{m_1m_2}] = 2i\sin\left[\frac{\pi}{2}(m_1n_2 - m_2n_1)\right] Y_{m_1+n_1,m_2+n_2} \tag{95}$$

reproduce exactly the $su(2)$ algebra

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ij}{}^k\sigma_k \tag{96}$$

Then, it results from (44) that the coefficients a^{mn} (Wigner densities) corresponding to σ_1, σ_2, and σ_3 are, respectively,

$$a_1 = 2\delta^{m1}\delta^{n0} \tag{97a}$$

$$a_2 = 2\delta^{m1}\delta^{n1} \tag{97b}$$

$$a_3 = 2\delta^{m0}\delta^{n1} \tag{97c}$$

The associated *symbols* are given by [see (87)]

$$f_1 = e^{i\pi s} \tag{98a}$$

$$f_2 = e^{i\pi(s-r)} \tag{98b}$$

$$f_3 = e^{-i\pi r} \tag{98c}$$

Using (80), we find the different twisted convolutions

$$a_1 \times_2 a_2 = 2\delta^{k2}\delta^{l1}e^{-i\pi l/2}, \qquad a_2 \times_2 a_1 = 2\delta^{k2}\delta^{l1}e^{i\pi(k-l)/2} \tag{99a}$$

$$a_2 \times_2 a_3 = 2\delta^{k1}\delta^{l2}e^{i\pi(k-l)/2}, \qquad a_3 \times_2 a_2 = 2\delta^{k1}\delta^{l2}e^{i\pi k/2} \tag{99b}$$

$$a_3 \times_2 a_1 = 2\delta^{k1}\delta^{l1}e^{i\pi k/2}, \qquad a_1 \times_2 a_3 = 2\delta^{k1}\delta^{l1}e^{-i\pi l/2} \tag{99c}$$

Using (90d), (87), and (41b), we find the different star-products

$$f_1 *_2 f_2 = if_3, \qquad f_2 *_2 f_1 = -if_3 \tag{100a}$$

$$f_2 *_2 f_3 = if_1, \qquad f_3 *_2 f_2 = -if_1 \tag{100b}$$

$$f_3 *_2 f_1 = if_2, \qquad f_1 *_2 f_3 = -if_2 \tag{100c}$$

Finally, the deformed brackets become [see (92) or (94c)]

$$\{f_i, f_j\}_2 = \frac{2}{\pi} \epsilon_{ij}{}^k f_k \tag{101}$$

It is easy to see that the bracket $\{\,,\}_2$ obeys the Jacobi identity (30).

We remark here that the symbols redefined as $F_i = i\pi f_i$ will satisfy the same $su(2)$ algebra with the deformed bracket $\{\,,\}_2$ as do their associated operators σ_i with the commutator $[\,,]$ [see (96)].

This simple example shows how the Schwinger basis $\{Y_{mn}\}$ describing a quantum lattice two-torus may be viewed as a set of generators of some noncommutative associative algebra equipped with a star-product and defined on the ordinary lattice two-torus.

In the general case, the relation (87) represents a Fourier expansion in the basis $\{\omega^{\vec{\mu}\times\xi} = \exp(i(2\pi/N)\vec{\mu} \times \vec{\xi})\}$ of functions on the lattice torus. From (73a), one may associate to a given Schwinger generator $Y_{\vec{\nu}}$ the Wigner density

$$a^{\vec{\mu}} = N^d \delta^{\vec{\nu}\vec{\mu}} = N^d \delta^{\vec{k}\vec{m}} \delta^{\vec{l}\vec{n}} = N^d \delta^{k_1 m_1} \cdots \delta^{k_d m_d} \cdot \delta^{l^1 n^1} \cdots \delta^{l^d n^d}$$

where $\vec{\mu} = (\vec{m}, \vec{n})$, $\vec{\nu} = (\vec{k}, \vec{l})$, $\vec{m} = (m_1, \ldots, m_d)$, $\vec{n} = (n^1, \ldots, n^d)$, $\vec{k} = (k_1, \ldots, k_d)$, and $\vec{l} = (l^1, \ldots, l^d)$. The associated symbol is then given by $f(\vec{\xi}) = \omega^{\vec{\rho}\times\xi}$. It is remarkable that the Wigner functions (symbols) corresponding to the basic Schwinger operators are just the Fourier basic functions. Moreover, to the product $Y_{\vec{\mu}} Y_{\vec{\nu}}$ of two generators of the Schwinger basis will correspond the Wigner density $c^{\vec{\rho}} = N^d \delta^{\vec{\nu}, \vec{\rho} - \vec{\mu}} e^{i\alpha_2[\vec{\rho};\vec{\mu}]}$ and, if we denote the symbols associated with $Y_{\vec{\mu}}$ and $Y_{\vec{\nu}}$ by f_μ and f_ν, respectively, we obtain the twisted products

$$f_\mu *_N f_\nu(\xi) = e^{i\alpha_2[\vec{\nu};\vec{\mu}]} \omega^{(\vec{\mu}+\vec{\nu})\times\xi} \tag{102}$$

and the deformed brackets

$$\{f_\mu(\vec{\xi}), f_\nu(\vec{\xi})\}_N = \frac{N}{\pi} \sin\left(\frac{\pi}{N} \vec{\nu} \times \vec{\mu}\right) \omega^{(\vec{\mu}+\vec{\nu})\times\xi} \tag{103}$$

Furthermore, one may deduce from (69) the following relation:

$$Y(\vec{c}, \vec{d}) \cdot Y(\vec{a}, \vec{b}) = q^{(\vec{a}\,\vec{b})\times(\vec{c}\,\vec{d})} Y(\vec{a}, \vec{b}) \cdot Y(\vec{c}, \vec{d}) \tag{104}$$

where

$$q = e^{i\hbar} \tag{105}$$

such that $q \to 1$ when $\hbar \to 0$. Choosing (Djemai, 1992)

$$\vec{a} = \vec{d} = \vec{u}, \qquad \vec{b} = \vec{c} = \vec{o} \tag{106a}$$

where $\vec{u}$ is some unit vector, we get

$$\Upsilon(\vec{o}, \vec{u}) \cdot \Upsilon(\vec{u}, \vec{o}) = q\Upsilon(\vec{u}, \vec{o}) \cdot \Upsilon(\vec{o}, \vec{u}) \tag{106b}$$

which coincides with the defining relation for the Manin plane (Manin, 1988).

In the discrete case, the formula analogous to (106b) is no other than the basic relation (32), with $\omega = e^{i2\pi/N} \to 1$ when $N \to \infty$. Finally, it is important to point out that the LQPS is a space with a *matrix structure* and this leads us to use the formalism of matrix differential geometry (Dubois-Violette, 1988, 1990; Dubois-Violette *et al.*, 1989a,b, 1990a,b).

5. MATRIX DIFFERENTIAL GEOMETRY

For a general review on the subject, refer to Djemai (n.d.-c). Here we present a brief summary of this formalism.

Let $M_N(C)$ be the algebra End(C) of all endomorphisms of C^N, i.e., the set of complex $N \times N$ matrices, $N \geq 2$. Then $M_N(C)$ is an associative noncommutative C*-algebra with unit 1. Let $\{E_k\}$, $k \in I = \{1, 2, \ldots, N^2 - 1\}$ be a basis of self-adjoint traceless $N \times N$ matrices. Then, $\{1, E_k\}$ is a convenient basis of $M_N(C)$ consisting of Hermitian matrices. One has the following multiplication table:

$$E_k \cdot E_l = K_{kl} 1 + \left(S_{kl}^m - \frac{i}{2} C_{kl}^m \right) E_m \tag{107}$$

where K_{kl} are the components of the Killing form of $su(N)$ given by

$$K_{kl} = K_{lk} = \frac{1}{N} \operatorname{Tr}[E_k \cdot E_l] \tag{108}$$

The coefficients C_{kl}^m correspond canonically to structure constants of $su(N)$, i.e.,

$$[iE_k, iE_l] = C_{kl}^m (iE_m) \tag{109}$$

Here, we are not interested in the notion of the manifold itself, but only in the algebra of *functions*. Actually, our *functions* are no other than the matrices 1, E_k, and C-linear combinations.

Let Der($M_N(C)$) be the algebra of all derivations of $M_N(C)$ in itself:

$$\operatorname{Der}(M_N(C)) = \{\chi \in \operatorname{End}(M_N(C)/\chi(E \cdot F) = \chi(E) \cdot F + E \cdot \chi(F), \quad \forall E, F \in M_N(C)\}$$

Since all the derivations of $M_N(C)$ are *inner*, it follows that the complex (resp. real) Lie algebra $\mathrm{Der}(M_N(C))$ [resp. $\mathrm{Der}_R(M_N(C))$] reduces to the Lie algebra $sl(N)$ [resp. $su(N)$]. The basis $\{e_k\}$ of all derivations of $M_N(C)$ is formed by the adjoint action of the generators $\{E_k\}$, $k \in I$, of $su(N) \equiv \mathrm{Der}_R(M_N(C))$. Then, there are only $N^2 - 1$ independent basic derivations e_k defined by

$$e_k = \mathrm{ad}(iE_k) \tag{110}$$

such that

$$[e_k, e_l] = C^m_{kl} e_m \tag{111}$$

$$e_k(iE_l) = [iE_k, iE_l] = C^m_{kl}(iE_m) \tag{112}$$

Then, any element χ of $\mathrm{Der}_R(M_N(C))$ will be written as

$$\chi = \chi^k e_k \tag{113}$$

and contrary to the commutative case, $\mathrm{Der}(M_N(C))$ does not form a $M_N(C)$-module, i.e., a derivation multiplied by a matrix is not a derivation.

Furthermore, it is shown that the smallest differential subalgebra $\Omega_{\mathrm{Der}}(M_N(C))$ of the complex $C(\mathrm{Der}(M_N(C)); M_N(C))$ which contains $M_N(C)$ is the complex itself.

Any p-form $\alpha_p \in \Omega^p_{\mathrm{Der}}(M_N(C))$ is a p-linear antisymmetric mapping:

$$\begin{aligned} \alpha_p\colon\quad & [\mathrm{Der}(M_N(C))]^p \to M_N(C) \\ & (\chi_1, \chi_2, \ldots, \chi_p) \to \alpha_p(\chi_1, \chi_2, \ldots, \chi_p) \end{aligned} \tag{114}$$

and its differential $d\alpha_p \in \Omega^{p+1}_{\mathrm{Der}}(M_N(C))$ is defined by

$$\begin{aligned} d\alpha_p(\chi_0, \chi_1, \ldots, \chi_p) &= \sum_{k=0}^{p} (-1)^k \chi_k \alpha_p(\chi_0, \ldots, \overset{\hat{k}}{.}, \ldots, \chi_p) \\ &+ \sum_{0 \le r \le s \le p} (-1)^{r+s} \alpha_p([\chi_r, \chi_s], \chi_0, \ldots, \overset{\hat{r}}{.}, \ldots, \overset{\hat{s}}{.}, \ldots, \chi_p) \end{aligned} \tag{115a}$$

for $\chi_0, \ldots, \chi_p \in \mathrm{Der}(M_N(C))$ and $\overset{\hat{k}}{.}$ means omission of χ_k such that

$$d^2 = 0 \tag{115b}$$

$$dE(\chi) = \chi(E) = \mathrm{ad}(iF)(E) = i[F, E] \tag{115c}$$

for any $E \in M_N(C)$ and $\chi = \mathrm{ad}(iF) \in \mathrm{Der}(M_N(C))$ with $F \in M_N(C)$ [see (110)].

The action of the inner product i_χ and the Lie derivative L_χ on p-forms α_p is defined as follows:

$$i_\chi \alpha_p(\chi_1, \ldots, \chi_{p-1}) = \alpha_p(\chi, \chi_1, \ldots, \chi_{p-1})/i_\chi(M_N(\mathrm{C})) = 0 \tag{116}$$

and

$$L_\chi = i_\chi d + d i_\chi \tag{117}$$

such that

$$L_\chi(E) = \chi(E) \qquad \text{for any} \quad E \in M_N(\mathrm{C}) \text{ and } \chi \in \mathrm{Der}(M_N(\mathrm{C})) \tag{118a}$$

$$\alpha_p \text{ is called } \textit{invariant} \text{ if } L_\chi(\alpha_p) = 0 \qquad \text{for } \chi \in \mathrm{Der}(M_N(\mathrm{C})) \tag{118b}$$

$$i_{\chi_1} i_{\chi_2} + i_{\chi_2} i_{\chi_1} = 0 \tag{118c}$$

$$L_{\chi_1} i_{\chi_2} - i_{\chi_2} L_{\chi_1} = i_{[\chi_1, \chi_2]} \tag{118d}$$

$$L_{\chi_1} L_{\chi_2} - L_{\chi_2} L_{\chi_1} = L_{[\chi_1, \chi_2]} \tag{118e}$$

Now, in order to construct the whole graded vector space $\Omega_{\mathrm{Der}}(M_N(\mathrm{C}))$ of *matrix* forms we use the differential d as defined by the relations (115) and the exterior product. Firstly, define the space $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$ of *matrix* 1-forms. Let $\{\theta^k\}$, $k \in I = \{1, 2, \ldots, N^2 - 1\}$, be a basis of 1-forms *dual* to the basis $\{e_m\}$, $m \in I$, of real derivations [see (110)], i.e.,

$$\theta^k(e_m) = \delta^k_m 1 \tag{119}$$

By definition, $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$ is a *module over* $M_N(\mathrm{C})$, i.e., one may also define the forms

$$E_m \theta^k = \theta^k E_m \tag{120}$$

such that [see (109), (110), and (115c)]

$$dE_k(e_j) = e_j(E_k) = i[E_j, E_k] = C^m_{jk} E_m \tag{121}$$

which means that [see (119)]

$$dE_k = -C^m_{kl} E_m \theta^l \tag{122}$$

This relation can be inverted to yield

$$\theta^k = -\frac{i}{N^2} K^{pq} K^{kr} E_p \cdot E_r dE_q \tag{123}$$

Now, the Grassmannian structure on $\Omega_{\mathrm{Der}}(M_n(\mathrm{C}))$ is introduced as usual by defining the exterior product on the basis $\{\theta^k\}$:

$$\theta^k \wedge \theta^m = -\theta^m \wedge \theta^k \tag{124}$$

Let us remark here that, in general, we could have chosen as a basis of 1-forms the set $\{dE_k\}$, $k \in I$, but the latter leads to problems due to the following noncommutativity properties:

$$E_k\,(dE_m) \neq (dE_m)\,E_k \tag{125a}$$

$$dE_k \wedge dE_m \neq -dE_m \wedge dE_k \tag{125b}$$

when the basis $\{\theta^k\}$ possesses the good properties (120) and (124).

Using the relation (115a), one obtains the important identity

$$d\theta^k = -\tfrac{1}{2}C^k_{pq}\theta^p \wedge \theta^q \tag{126}$$

which is the analog of the Maurer–Cartan identity on the group manifolds.

The relations (107), (120), (122), (124), and (126) give a *presentation* of $\Omega_{\mathrm{Der}}(M_N(\mathrm{C}))$ *associated to the basis* $\{E_k\}$.

The element θ of $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$ defined by

$$\theta = E_k\theta^k \tag{127}$$

is independent of the choice of the E_k. In fact, one has

$$\theta(\mathrm{ad}(iE)) = E - \frac{1}{N}\,\mathrm{Tr}(E)\;1 \tag{128}$$

with $E \in M_N(\mathrm{C})$. Furthermore, θ is *invariant* and any invariant element of $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$ is a scalar multiple of θ. This 1-form is called the *canonical invariant element* of $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$. Using it, we obtain for equations (122) and (126)

$$dE = i[\theta, E] \qquad \text{for any} \quad E \in M_N(\mathrm{C}) \tag{129}$$

$$d(-i\theta) + (-i\theta)^2 = 0 \tag{130}$$

respectively.

Many more things which we shall not need here may be introduced on matrix spaces, such as involution, integration of p-forms, canonical Riemannian structure for $M_N(\mathrm{C})$, Hodge-star operator, coderivative, Laplace–Beltrami operator, Hodge–De Rham decomposition, connections and their curvatures, etc.

For instance, the integral of an $(N^2 - 1)$-form may be defined using the trace,

$$\int E\theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^{N^2-1} = \mathrm{Tr}(E) \tag{131}$$

We conclude this section by presenting very succinctly the example of $N = 2$, where the basis of $M_2(C)$ is formed by the 2×2 unit matrix 1 and the Hermitian traceless 2×2 Pauli matrices σ_j, $j = 1, 2, 3$ [see (59)].

In this case, one has

$$\sigma_k \cdot \sigma_l = \delta_{kl} 1 + i\epsilon_{kl}^m \sigma_m$$

$$K_{kl} = \delta_{kl}, \qquad C_{kl}^m = C_{klm} = -2\epsilon_{klm}$$

$$S_{kl}^m = 0, \qquad e_k = \mathrm{ad}(i\sigma_k), \qquad \theta^k(e_p) = \delta_p^k 1 \tag{132}$$

and

$$\sigma_k \theta^l = \theta^l \sigma_k \tag{133}$$

$$\theta^k \wedge \theta^l = -\theta^l \wedge \theta^k \tag{134}$$

$$d\sigma_k = 2\epsilon_{kl}^m \sigma_m \theta^l \tag{135}$$

$$d\theta^k = \epsilon_{pq}^k \theta^p \wedge \theta^q \tag{136}$$

The relations (132)–(136) give a *presentation* of $\Omega_{\mathrm{Der}}(M_2(C))$ associated with the basis $\{\sigma_j\}$.

6. QUANTUM (MATRIX) SYMPLECTIC FORMALISM

Our approach consists in formulating the Weyl–Schwinger realization of the Heisenberg group (see Section 4) in a matrix context and adapting it to the formalism of the matrix differential symplectic geometry presented in Section 5.

The Schwinger basis $\{Y_{mn}/m, n = 0, 1, \ldots, N - 1\}$ given by equation (39) may be considered as a basis of the matrix algebra $M_N(C)$ consisting of Hermitian matrices. Excluding the unit matrix Y_{00}, the Y_{mn}s are $N^2 - 1$ unitary traceless $N \times N$ matrices and may be viewed as a basis for $su(N)$.

Then, the relation (44) gives the expansion of any element of $M_N(C)$ in the Schwinger basis $\{Y_{mn}\}$. The multiplication table of these basic elements is given by the generalized composition law (51). Here the numbers m and n are defined modulo N. From now, we will use the compact notation given by equations (49).

Let us first define the Lie algebra of derivations of $M_N(C)$ relative to the basis $\{Y_{\vec{m}}/\vec{m} \neq \vec{0}\}$. Define the basic derivations as given by equations (110) and (112), i.e.,

$$e_{\vec{m}} = \mathrm{ad}(Y_{\vec{m}})/e_{\vec{m}}(Y_{\vec{p}}) = [Y_{\vec{m}}, Y_{\vec{p}}] \tag{137}$$

Using

$$[Y_{\vec{m}}, Y_{\vec{n}}] = -2i \sin\{\alpha_2(\vec{m}, \vec{n})\}\ Y_{\vec{m}+\vec{n}} \tag{138}$$

and the Jacobi identity

$$[Y_{\vec{m}}, [Y_{\vec{n}}, Y_{\vec{p}}]] + \text{o.p.} = 0 \tag{139}$$

one easily find that the basic derivations $e_{\vec{m}}$ obey the following relation:

$$[e_{\vec{m}}, e_{\vec{n}}] = -2i \sin\{\alpha_2(\vec{m}, \vec{n})\} e_{\vec{m}+\vec{n}} \tag{140}$$

Comparing with equation (111), we get the following structure constants:

$$C_{\vec{m}\vec{n}}^{\vec{p}} = -2i \sin\{\alpha_2(\vec{m}, \vec{n})\}\ \delta_{\vec{m}+\vec{n}}^{\vec{p}} \tag{141}$$

where the Kronecker symbol is defined by equation (50e).

We will now compute the Killing metric for $su(N)$ relative to the basis $\{Y_{\vec{m}}/\vec{m} \neq \vec{0}\}$ by using

$$\text{Tr}[\text{ad}(A)\cdot\text{ad}(B)] = 2N\ \text{Tr}[A\cdot B] = 2 \sum_{\vec{m}\,\vec{n}} K_{\vec{m}\vec{n}} a^{\vec{m}} b^{\vec{n}} \tag{142}$$

where $A, B \in M_N(\mathrm{C})$. It results from a direct computation that $K_{\vec{m}\vec{n}}$ is given by

$$K_{\vec{m}\vec{n}} = \frac{1}{2N^2} \sum_{\vec{r},\vec{s}} C_{\vec{m}\vec{r}}^{\vec{s}} C_{\vec{n}\vec{s}}^{\vec{r}} \tag{143}$$

Using equation (108), i.e.,

$$K_{\vec{m}\vec{n}} = \frac{1}{N} \text{Tr}[Y_{\vec{m}} Y_{\vec{n}}] \tag{144}$$

the relation (138), and the property (46a), i.e.,

$$\frac{1}{N} \text{Tr}[Y_{\vec{m}}] = \delta_{\vec{m}\vec{0}} \tag{145}$$

we find that the Killing metric is given by

$$K_{\vec{m}\vec{n}} = \delta_{\vec{m}+\vec{n}\,\vec{0}} \tag{146}$$

Remark that the metric $g_{\vec{m}\vec{n}}$ defined on the operator algebra by equation (48) is now tied to the Killing metric by the following relation:

$$g_{\vec{m}\vec{n}} = K_{\vec{m},-\vec{n}} = \delta_{\vec{m}\vec{n}} \tag{147}$$

for $\vec{m}, \vec{n} \neq \vec{0}$.

It remains to determine the general expression of the symmetric quantity $S_{\vec{m}\vec{n}}^{\vec{p}}$. Using the relation

$$\tfrac{1}{2}[Y_{\vec{m}}\cdot Y_{\vec{n}} + Y_{\vec{n}}\cdot Y_{\vec{m}}] = K_{\vec{m}\vec{n}}1 + S_{\vec{m}\vec{n}}^{\vec{p}} Y_{\vec{p}} = \cos\{\alpha_2(\vec{m}, \vec{n})\} Y_{\vec{m}+\vec{n}} \tag{148}$$

it follows that all the quantities $S_{\vec{m}\vec{n}}^{\vec{p}}$ identically vanish:

$$S_{\vec{m}\vec{n}}^{\vec{p}} = -2\delta_{\vec{m}+\vec{n},\vec{0}}\delta^{\vec{p},\vec{0}} \equiv 0 \tag{149}$$

since the indices $\vec{m}$, $\vec{n}$, and $\vec{p}$ are defined such that the vanishing index $\vec{0}$ is excluded.

We finally obtain the following multiplication table:

$$\begin{aligned} Y_{\vec{m}} \cdot Y_{\vec{n}} &= \exp[i\alpha_2(\vec{n}, \vec{m})] Y_{\vec{m}+\vec{n}} \\ &= K_{\vec{m}\vec{n}} 1 + (S_{\vec{m}\vec{n}}^{\vec{p}} + \tfrac{1}{2} C_{\vec{m}\vec{n}}^{\vec{p}}) Y_{\vec{p}} \\ &= \delta_{\vec{m}+\vec{n},\vec{0}} 1 - i \sin\{\alpha_2(\vec{m}, \vec{n})\}\ \delta_{\vec{m}+\vec{n}}^{\vec{p}} Y_{\vec{p}} \end{aligned} \tag{150}$$

Let us introduce now a basis of 1-forms, denoted by $\theta^{\vec{m}}$, dual to the basis of derivations $\{e_{\vec{n}}\}$ [see the relation (119)]. Of course, the indices $\vec{m}$ and $\vec{n}$ are always defined such that the value $\vec{0}$ is excluded. Then, we have

$$\theta^{\vec{m}}(e_{\vec{n}}) = \delta_{\vec{n}}^{\vec{m}} 1 \tag{151}$$

such that the $\theta^{\vec{m}}$ satisfy the following relation:

$$\theta^{\vec{m}} \wedge \theta^{\vec{n}} = -\theta^{\vec{n}} \wedge \theta^{\vec{m}} \tag{152}$$

Since $\Omega_{\mathrm{Der}}^1(M_N(\mathrm{C}))$ is an $M_N(\mathrm{C})$-module, one also has

$$Y_{\vec{m}}\theta^{\vec{n}} = \theta^{\vec{n}} Y_{\vec{m}} \tag{153}$$

It is also easy to see that

$$\begin{aligned} dY_{\vec{m}}(e_{\vec{n}}) &= e_{\vec{n}}(Y_{\vec{m}}) = [Y_{\vec{n}}, Y_{\vec{m}}] \\ &= -2i \sin\{\alpha_2(\vec{n}, \vec{m})\}\ Y_{\vec{m}+\vec{n}} = C_{\vec{n}\vec{m}}^{\vec{p}} Y_{\vec{p}} \end{aligned} \tag{154}$$

so that

$$dY_{\vec{m}} = -\sum_{\vec{n},\vec{p}} C_{\vec{m}\vec{n}}^{\vec{p}} Y_{\vec{p}} \theta^{\vec{n}} \tag{155}$$

Using equations (124), (115a), and (141), one finds

$$\begin{aligned} d\theta^{\vec{m}} &= -\frac{1}{2} \sum_{\vec{p},\vec{q}} C_{\vec{p}\vec{q}}^{\vec{m}} \theta^{\vec{p}} \wedge \theta^{\vec{q}} \\ &= -i \sum_{\vec{q}} \sin\{\alpha_2(\vec{m}, \vec{q})\}\ \theta^{\vec{q}} \wedge \theta^{\vec{m}-\vec{q}} \end{aligned} \tag{156}$$

Then, equations (150), (152), (153), (155), and (156) give a presentation of $\Omega_{\mathrm{Der}}(M_N(\mathrm{C}))$ associated to the Schwinger basis $\{Y_{\vec{m}} / \vec{m} \neq \vec{0}\}$.

The element $\theta \in \Omega_{\mathrm{Der}}^1(M_N(\mathrm{C}))$ defined by

$$\theta = \sum_{\vec{m} \neq \vec{0}} Y_{\vec{m}} \theta^{\vec{m}} \tag{157}$$

will be called the *canonical* 1-*form.* As expected, this 1-form plays the role of the Liouville 1-form and its differential, the 2-form

$$\Omega = -d\theta \tag{158}$$

will represent the *quantum* (*matrix*) *symplectic* 2-*form.* It is given by

$$\begin{aligned}\Omega &\equiv \tfrac{1}{2}\Omega_{\vec{m}\vec{n}}\theta^{\vec{m}} \wedge \theta^{\vec{n}} = -\tfrac{1}{2}C^{\vec{p}}_{\vec{m}\vec{n}}\, Y_{\vec{p}}\theta^{\vec{m}} \wedge \theta^{\vec{n}} = Y_{\vec{m}} d\theta^{\vec{m}} \\ &= i \sum_{\vec{m},\vec{n}} \sin\{\alpha_2(\vec{m}, \vec{n})\} Y_{\vec{m}+\vec{n}}\theta^{\vec{m}} \wedge \theta^{\vec{n}}\end{aligned} \tag{159}$$

Then, the *quantum symplectic matrix* $\Omega_{\vec{m}\vec{n}}$ is an $(N^2 - 1) \times (N^2 - 1)$ antisymmetric hypermatrix whose entries are the $N \times N$ Schwinger matrices:

$$\Omega_{\vec{m}\vec{n}} = -C^{\vec{p}}_{\vec{m}\vec{n}} Y_{\vec{p}} = [Y_{\vec{n}}, Y_{\vec{m}}] = e_{\vec{n}}(Y_{\vec{m}}) = -\Omega_{\vec{n}\vec{m}} \tag{160}$$

Using equations (160) and (50c), we find the following relations:

$$e_{\vec{m}}(A) = \frac{-1}{N} \sum_{\vec{n}} \Omega_{\vec{m}\vec{n}} a^{\vec{n}} \tag{161}$$

$$[A, B] = \frac{-1}{N^2} \sum_{\vec{m},\vec{n}} a^{\vec{m}} \Omega_{\vec{m}\vec{n}} b^{\vec{n}} \tag{162}$$

$$dY_{\vec{m}} = \sum_{\vec{n}} \Omega_{\vec{m}\vec{n}} \theta^{\vec{n}} \tag{163}$$

$$dA = \frac{1}{N} \sum_{\vec{m},\vec{n}} a^{\vec{m}} \Omega_{\vec{m}\vec{n}} \theta^{\vec{n}} \tag{164}$$

The hypermatrix $\Omega_{\vec{m}\vec{n}}$ has an inverse which is defined as a hypermatrix $\Lambda^{\vec{m}\vec{n}}$ satisfying the following properties:

$$\Omega_{\vec{m}\vec{n}} \Lambda^{\vec{n}\vec{p}} = \Lambda^{\vec{p}\vec{n}} \Omega_{\vec{n}\vec{m}} = \delta^{\vec{p}}_{\vec{m}} 1_{N\times N} \tag{165}$$

In order to determine it, one must first define the inverses $Y^{\vec{m}}$ and $K^{\vec{m}\vec{n}}$ of the Schwinger matrix $Y_{\vec{m}}$ and the Killing metric $K_{\vec{m}\vec{n}}$, respectively.

The Killing metric, which may be used to raise or to lower indices, has an inverse defined by

$$K^{\vec{m}\vec{n}} K_{\vec{n}\vec{p}} = K_{\vec{p}\vec{n}} K^{\vec{n}\vec{m}} = \delta^{\vec{m}}_{\vec{p}} \tag{166a}$$

$$K^{\vec{m}\vec{n}} = \delta^{\vec{m}+\vec{n},\vec{0}} \tag{166b}$$

and the inverse $Y^{\vec{m}}$ is defined such that

$$Y^{\vec{m}} = K^{\vec{m}\vec{n}} Y_{\vec{n}} \qquad \text{and} \qquad Y_{\vec{m}} = K_{\vec{m}\vec{n}} Y^{\vec{n}} \tag{167}$$

so that the $Y^{\vec{m}}$ are just the Hermitian conjugates of $Y_{\vec{m}}$ [see equation (40)]:

$$Y^{\vec{m}} \equiv Y_{-\vec{m}} = Y_{\vec{m}}^{\pm} = Y_{\vec{m}}^{-1} \tag{168a}$$

$$Y_{\vec{m}} \equiv Y^{-\vec{m}} = (Y^{\vec{m}})^{+} = (Y^{\vec{m}})^{-1} \tag{168b}$$

Let us remark that equation (60c) may be rewritten in this context as

$$\sum_{\vec{m}\neq\vec{0}} \exp[i\alpha_2(\vec{m}, \vec{n})] = N^2\delta_{\vec{n}\vec{0}} - 1 \tag{169}$$

Using equations (143) and (150), one verifies that the hypermatrix $\Lambda^{\vec{m}\vec{n}}$ that obeys equation (165) must be of the form

$$\Lambda^{\vec{m}\vec{n}} = \frac{1}{N^2} Y^{\vec{n}} \cdot Y^{\vec{m}} = \frac{1}{N^2} K^{\vec{n}\vec{r}} K^{\vec{m}\vec{s}} Y_{\vec{r}} \cdot Y_{\vec{s}} \tag{170}$$

Notice that this hypermatrix is not antisymmetric. Indeed, one has

$$\Lambda^{\vec{n}\vec{m}} = \exp[2i\alpha_2(\vec{n}, \vec{m})]\, \Lambda^{\vec{m}\vec{n}} \tag{171}$$

Equation (155) [or (163)] can be inverted to yield [see equation (123)]

$$\theta^{\vec{n}} = \Lambda^{\vec{n}\vec{m}} dY_{\vec{m}} = \frac{1}{N^2} Y^{\vec{m}} \cdot Y^{\vec{n}} dY_{\vec{m}} = \frac{1}{N^2} K^{\vec{m}\vec{r}} K^{\vec{n}\vec{s}} Y_{\vec{r}} \cdot Y_{\vec{s}} dY_{\vec{m}} \tag{172}$$

Finally, if we consider $N = 2$, with $\vec{1} = (1, 0)$, $\vec{2} = (1, 1)$, and $\vec{3} = (0, 1)$, we obtain

$$\Omega_{mn} = 2i\begin{pmatrix} 0 & \sigma_3 & -\sigma_2 \\ -\sigma_3 & 0 & \sigma_1 \\ \sigma_2 & -\sigma_1 & 0 \end{pmatrix}, \qquad \Lambda^{mn} = \frac{1}{4}\begin{pmatrix} 1 & -i\sigma_3 & i\sigma_2 \\ i\sigma_3 & 1 & -i\sigma_1 \\ -i\sigma_2 & i\sigma_1 & 1 \end{pmatrix} \tag{173}$$

Furthermore, one may also verify that the *quantum canonical* 1-form θ defined by (157) is effectively the basic *invariant element* of $\Omega^1_{\mathrm{Der}}(M_N(\mathrm{C}))$.

Indeed, for any vector field $\chi \in \mathrm{Der}(M_N(\mathrm{C}))$ given by

$$\chi = \frac{1}{N} \sum_{\vec{m}} \chi^{\vec{m}} e_{\vec{m}} \tag{174}$$

one has

$$i_\chi(\theta) = \theta(\chi) = \frac{1}{N} \sum_{\vec{m}} \chi^{\vec{m}} Y_{\vec{m}} \tag{175a}$$

$$\begin{aligned} di_\chi(\theta) &= d[\theta(\chi)] = \frac{1}{N} \sum_{\vec{m}} \chi^{\vec{m}} dY_{\vec{m}} \\ &= \frac{1}{N} \sum_{\vec{m}\vec{n}} \chi^{\vec{m}} \Omega_{\vec{m}\vec{n}} \theta^{\vec{n}} = \frac{-1}{N} \sum_{\vec{m}\vec{n}\vec{p}} \chi^{\vec{m}} C^{\vec{p}}_{\vec{m}\vec{n}} Y_{\vec{p}} \theta^{\vec{n}} \end{aligned} \tag{175b}$$

and

$$i_\chi d(\theta) = -i_\chi(\Omega) = -\tfrac{1}{2}\sum_{\vec{m}} \Omega_{\vec{m}\vec{n}} i_\chi(\theta^{\vec{m}} \wedge \theta^{\vec{n}})$$

$$= \frac{-1}{N} \sum_{\vec{m},\vec{n}} \chi^{\vec{m}} \Omega_{\vec{m}\vec{n}} \theta^{\vec{n}} = \frac{1}{N} \sum_{\vec{m},\vec{n},\vec{p}} \chi^{\vec{m}} C^{\vec{p}}_{\vec{m}\vec{n}} Y_{\vec{p}} \theta^{\vec{n}} \tag{175c}$$

so that, for any $\chi \in \mathrm{Der}(M_N(\mathbb{C}))$, one has

$$L_\chi(\theta) = [i_\chi d + d i_\chi](\theta) = 0 \tag{176}$$

Moreover, it is easy to see that

$$L_\chi(\Omega) = -L_\chi(d\theta) = -d(L_\chi(\theta)) = 0 \tag{177}$$

and consequently all vector fields χ are said in the classical terminology to be *strictly Hamiltonian* or *globally Hamiltonian* (Guillemin and Sternberg, 1984; Abraham and Marsden, 1985; Arnold, 1989). This means that to each vector $\chi \in \mathrm{Der}(M_N(\mathbb{C}))$ will correspond an element $\alpha = dA$ of $\Omega^1_{\mathrm{Der}}(M_N(\mathbb{C}))$ such that equation (177) holds, and conversely, to each 1-form dA one associates a Hamiltonian vector field χ_A such that $i_{\chi_A}\Omega = dA$, where the components $\chi_A^{\vec{m}}$ of χ_A in the basis $\{e_{\vec{m}}\}$ coincide with the coefficients $a^{\vec{m}}$ of A in the Schwinger basis.

Hence, operators of the form $\theta(\chi)$ play the role of *generating functions*:

$$d[\theta(\chi_1)](\chi_2) = \Omega(\chi_1, \chi_2) = -d[\theta(\chi_2)](\chi_1) \tag{178}$$

for any $\chi_1, \chi_2 \in \mathrm{Der}(M_N(\mathbb{C}))$.

Equation (137) was chosen rather than equation (110) to be in conformity with the results given by equations (91c) and (91e) for $d = 1$, i.e.,

$$[A, B] = i\frac{2\pi}{N} C = \frac{2\pi i}{N}\frac{1}{N}\sum_{\vec{r}} c^{\vec{r}} Y_{\vec{r}}$$

$$\Rightarrow c^{\vec{r}} = \frac{N}{\pi}\frac{1}{N}\sum_{\vec{m}} a^{\vec{m}} b^{\vec{r}-\vec{m}} \sin[\alpha_2(\vec{r}, \vec{m})] \tag{179}$$

In analogy with the classical case, the corresponding *quantum* (*matrix*) *Poisson bracket* is then defined by

$$\{A, B\}_m =: \Omega(\chi_B, \chi_A) = \chi_A(B) = -\chi_B(A) \tag{180}$$

As expected, this *matrix Poisson bracket* is just the commutator

$$\{A, B\}_m = [A, B] = -\frac{1}{N^2}\sum_{\vec{m},\vec{n}} a^{\vec{m}} \Omega_{\vec{m}\vec{n}} b^{\vec{n}}$$

$$= i\frac{2\pi}{N} C = \frac{2i}{N^2}\sum_{\vec{m},\vec{r}} a^{\vec{m}} b^{\vec{r}-\vec{m}} \sin[\alpha_2(\vec{r}, \vec{m})] Y_{\vec{r}} \tag{181}$$

and we can read the *discrete version of the Moyal bracket components* [see equations (91c) and (91d)].

Thus, the *quantum (matrix) symplectic 2-form* Ω defined in the context of the *noncommutative differential geometry of matrix algebras* $M_N(\mathbb{C})$ gives directly the *discrete version of the Moyal bracket.*

In other respects, it is easy to verify that the Jacobi identity

$$[A, [B, C]] + c \cdot p = 0 \tag{182}$$

is equivalent to the relation

$$\begin{aligned}[\chi_A, \chi_B](C) &= \chi_{\{A,B\}_m}(C) = -\chi_C(\{A, B\}_m) \\ &= dC(\chi_{\{A,B\}_m}) = -d(\{A, B\}_m)(\chi_C)\end{aligned} \tag{183}$$

for all operators A, B, and C. In particular, we have

$$\begin{aligned}[e_{\vec{m}}, e_{\vec{n}}](Y_{\vec{q}}) &= -e_{\vec{q}}(\{Y_{\vec{m}}, Y_{\vec{n}}\}_m) \\ &= dY_{\vec{q}}([e_{\vec{m}}, e_{\vec{n}}]) = -d(\{Y_{\vec{m}}, Y_{\vec{n}}\}_m)(e_{\vec{q}})\end{aligned} \tag{184}$$

We deduce from equation (183) that the *matrix Poisson bracket* plays the role of the generating function of the Lie bracket of the corresponding Hamiltonian vector fields.

One may also introduce the antisymmetric hypermatrix $\Omega^{\vec{m}\vec{n}}$ defined by

$$\Omega^{\vec{m}\vec{n}} = K^{\vec{m}\vec{r}} K^{\vec{n}\vec{s}} \Omega_{\vec{r}\vec{s}} = [Y^{\vec{n}}, Y^{\vec{m}}] = N^2(\Lambda^{\vec{m}\vec{n}} - \Lambda^{\vec{n}\vec{m}}) = -\Omega^{\vec{n}\vec{m}} \tag{185}$$

In order to complete the analogy with the classical case, we give here the quantum analogs of the classical relations (6)–(7), (9), and (12)–(13), respectively:

$$\chi_A = \frac{1}{N} \sum_{\vec{m}} \chi_A^{\vec{m}} e_{\vec{m}}, \qquad \chi_A^{\vec{m}} = a^{\vec{m}} = e_{\vec{n}}(A) \Lambda^{\vec{n}\vec{m}} \tag{186}$$

$$\begin{aligned}\{A, B\}_m &= [A, B] = -\Omega(\chi_A, \chi_B) \\ &= \chi_A(B) = -\chi_B(A) = e_{\vec{m}}(A) \Lambda^{\vec{m}\vec{n}} e_{\vec{n}}(B)\end{aligned} \tag{187}$$

and

$$\{A, B \cdot C\}_m = \{A, B\}_m \cdot C + B \cdot \{A, C\}_m \underset{\text{rule}}{\overset{\text{Leibnitz}}{\Rightarrow}}$$

$$\chi_A(B \cdot C) = \chi_A(B) \cdot C + B \cdot \chi_A(C) \tag{188}$$

when the quantum analogs of equations (5) and (10)–(11) are already given by equations (159) and (182)–(183), respectively.

7. DISCUSSION AND PERSPECTIVES

In this work, we have emphasized the importance of the particular choice of the Schwinger basis in the discrete realization of the Heisenberg group and in the construction of the associated discrete Weyl–Wigner–Moyal and Hamiltonian formalisms. Quantum mechanics is then presented as a *matrix symplectic geometry*.

Moreover, it has been shown (Landsman, 1992) that the discrete Weyl–Heisenberg algebra may give some good understanding of the notion of the quantum phase space, this latter being necessary to completely formulate quantum mechanics.

Furthermore, we find in the literature many attempts to describe quantum mechanics in different directions. Essentially, the tools used for this program are the star-deformation, quantum algebras, and noncommutative differential geometry; see, for instance, Dubois-Violette (1988, 1990), Dubois-Violette *et al.* (1989a,b, 1990a,b), Landsman (1992), Flato and Lu (1991), Flato and Sternheimer (1991), Lu (1992), Dimakis and Muller-Hoissen (1992), and Majid (1992).

In this context, quantum mechanics is presented in Dimakis and Muller-Hoissen (1992) as a noncommutative symplectic geometry and in Majid (1992) as a quantum double.

It has also been argued that it is possible to describe classical and quantum mechanics in a unified scheme of noncommutative geometry (Dass, 1994).

More generally, models of gauge theories based on quantum groups have been studied (Aref'eva and Volovich, 1990; Castellani, 1992; Brzezinski and Majid, 1992; Castellani and Montelro, 1993), and quantum versions of the Killing form and the structure constants presented.

The Newtonian and Lagrangian formulations of quantum mechanics in this context have also been treated (Sudberg, n.d.; Lukin *et al.*, 1993; Malik, 1993).

Furthermore, there is a deep link between the theory of quantum groups and the star-deformation. Effectively, Dubois-Violette (1990) showed that the C*-Hopf algebra of representative elements corresponding to a compact matrix quantum group (Woronowicz, 1987) is isomorphic as C*-coalgebra to the C*-algebra of representative functions on the corresponding classical compact group. This isomorphism appears as a generalization of the Weyl correspondence (Weyl, 1931).

In other respects, the question of the determination of the quantum symplectic structure in its discrete and continuous forms requires more investigation, although Dubois-Violette (1988, 1990) showed that the Heisenberg algebra $A_\hbar$ defined as a C*-algebra with unit generated by two Hermitian elements $\mathbf{p}$ and $\mathbf{q}$ such that $[\mathbf{q}, \mathbf{p}] = i\hbar$ correspond to the two-dimensional quantum phase space with a noncommutative symplectic structure given by

$$\omega = \sum_n \frac{1}{(i\hbar)^n(n+1)!} \underbrace{[\ldots[d\mathbf{p}, \mathbf{p}], \mathbf{p}], \ldots, \mathbf{p}]}_{n \text{ times}} \wedge \underbrace{[\ldots[d\mathbf{q}, \mathbf{q}], \mathbf{q}], \ldots, \mathbf{q}]}_{n \text{ times}}$$

which reduces to the ordinary one, $dp \wedge dq$, at the commutative limit $\hbar \to 0$. This is another confirmation that quantum mechanics can be understood as a noncommutative symplectic geometry.

In any case, it is clear that all these points and related topics, such as knot theory, braided groups and algebras, quantum fiber bundles, etc., constitute research directions of interest.

Finally, this work has been conceived as a step in a large program whose main aim is to give a more precise description of quantum mechanics in the framework of modern theories such as quantum groups, knot theory, noncommutative differential geometry, etc. (Djemai, 1994, n.d.-a,b).

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Erratum

Quantum Mechanics as a Matrix Symplectic Geometry

A. E. F. Djemai

The following corrections should be made to the above article that was published in *International Journal of Theoretical Physics*, **35**, 519–556 (1996).

The top of page 548 should read:

It follows that the quantities $S^{\vec{\ell}}_{\vec{m}\vec{n}}$ are given by:

$$S^{\vec{\ell}}_{\vec{m}\vec{n}} = \exp[i\alpha_2(\vec{m} + \vec{n}, \vec{p})]\cos[i\alpha_2(\vec{m}, \vec{n})]\Upsilon_{\vec{m}+\vec{n}-\vec{p}} - \delta_{\vec{m}+\vec{n},\vec{0}}\Upsilon_{-\vec{p}} \tag{149}$$

Finally, we have the following multiplication law:

$$\begin{aligned}\Upsilon_{\vec{m}} \cdot \Upsilon_{\vec{n}} &= \exp[i\alpha_2(\vec{n}, \vec{m})]\Upsilon_{\vec{m}+\vec{n}} \\ &= K_{\vec{m}\vec{n}}1 + (S^{\vec{\ell}}_{\vec{m}\vec{n}} + 1/2C^{\vec{\ell}}_{\vec{m}\vec{n}})\Upsilon_{\vec{p}}\end{aligned} \tag{150}$$